Fokker-Planck approach to the dynamics of mismatched charged-particle beams

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(Received 7 April 1994)

A semianalytic formalism is constructed for investigating the transverse dynamics of intense, mismatched charged-particle beams which are centered on, and propagate through, focusing channels. It uses the Fokker-Planck equation to account for the rapid evolution of the coarse-grained distribution function in the phase space of a single beam particle. It also incorporates the space-charge potential, which is calculated from Poisson's equation using the coarse-grained density. A simple phenomenological model of dynamical friction and diffusion represents the effects of turbulence triggered by charge redistribution. Sheet beams and fully two-dimensional beams are both considered in detail. In addition, closed-form solutions are presented for beams in which the space charge is negligible and noise arises from other stochastic processes.

PACS number(s): 41.85.Ja, 29.27.Bd, 41.75.−i, 52.35.Ra

I. INTRODUCTION

Space-charge forces in high-intensity linear accelerators are known to complicate the beam dynamics significantly. If at injection the beam is mismatched, charge redistribution takes place. This injects free energy associated with the mismatch in root-mean-square (rms) beam size, i.e., the "rms mismatch," into a spectrum of collective modes. If the free-energy density is large compared to the thermal-energy density, the mode amplitudes will also be large. Wave breaking will occur in phase space and the beam will become turbulent. As individual particles interact with the time-dependent mean global field and with clumps of particles within it, their orbits can be chaotic. The ensemble of particle orbits tends to cover the accessible phase space in a coarse-grained manner much more rapidly than binary encounters alone would permit and its range of energies widens. In this sense, the collective interactions cause rapid, collisionless relaxation toward a stationary state which leads to quasithermalization of the free energy. This scenario, which has been seen in both laboratory and numerical experiments [1–4], results in particles being ejected into high-amplitude orbits, causing the emittance to grow and a halo to form. The halo is of particular concern in linear accelerators envisioned for long-term continuous-wave operation because too much halo impingement on the accelerating structures would cause enough radioactivation to inhibit routine maintenance.

This sequence of events is akin to the collisionless relaxation of stellar systems in which the gravitational field is rapidly changing, a process known as "violent relaxation" [5–7]. It is clear that, after a sufficiently long time, interactions between particles, e.g., "collisions," will drive a system of stars, or of confined charged particles, to *the* steady state having the largest entropy. This state corresponds to thermodynamic equilibrium, for which the distribution function is the Maxwell-Boltzmann distribution [8]. However, the relaxation time associated with collisional encounters is much longer than the age of the universe in the case of galaxies and the beam's transit time in the case of linear accelerators. These many-body systems are therefore effectively *collisionless*, and the fact that they rapidly relax to a state of quasiequilibrium with a core-halo structure is surprising at first glance.

There are significant differences between the evolution of stellar systems and charged-particle beams, however. They arise because, whereas a stellar system is self-gravitating and (ideally) isolated, a beam is self-repelling and confined by an externally imposed focusing force. One consequence surfaces in connection with the state of thermodynamic equilibrium itself. For a self-gravitating system, collisional relaxation moves the system through a succession of configurations with progressively greater core concentration and lower total mass of bound stars. The maximum-entropy configuration therefore has a tightly bound central core of stars surrounded by a diffuse, and infinite, halo. There is no finite self-gravitating system for which the classical entropy is extremized [6]. By contrast, external focusing confines a beam for all time, provided none of its constituent particles impinges on the accelerator walls. In this sense, and unlike a stellar system, a beam can therefore seek a bound state of maximum entropy. This state is observable provided the beam is retained for a time which is long enough for collisional relaxation to run its course, as in storage rings, for example.

Another consequence of the difference between stellar systems and beams arises in conjunction with the detailed dynamics of violent relaxation. The size of a newly formed stellar system has an upper bound corresponding to the Jean's length λ_J. Collective modes with wavelengths greater than λ_J are unstable. Modes with wavelengths close to λ_J are soft, may easily be excited to large amplitudes, and can then dominate the violent relaxation of the stellar system [9]. In a beam there is no counterpart to λ_J. Rather the length scale of interest is the Debye length λ_D. If linear Landau damping were the only mechanism available to damp the mode spectrum, then fluctuations with wavelength $\lambda < \lambda_D$ would damp on a time scale of the order of a plasma period and fluctuations with $\lambda > \lambda_D$ would be relatively long-lived [10]. Beams that are space-charge dominated have radii exceeding λ_D and can therefore sustain these long-wavelength collective modes.

It is not obvious that violent relaxation drives a system of stars or charged particles all the way to thermodynamic equilibrium. The process halts when the potential stops changing and there is no guarantee that the free energy is fully equilibrated at that time. In the case of stellar systems, there is considerable evidence that not all "final" states are equally probable and an attempt has been made to formulate a maximum-entropy principle which excludes inaccessible states [6]. The same is probably also true for charged-particle beams. Both the laboratory and numerical experiments performed to date show that the density profiles of beams which have undergone violent relaxation possess an essentially thermalized core surrounded by a halo which is not fully thermalized [3,4,11].

We recently introduced a formalism to account for the complex transient dynamics of an intense, mismatched charged-particle beam and applied it to sheet beams [12]. Our purpose here is to elaborate the ideas and generalize the formalism presented in the earlier paper to make it applicable to beams of practical interest. In doing so, we endeavor to account for the dynamics of the bulk of the beam, i.e., that pertaining to the inner two-to-three rms radii. Thus although we recognize that violent relaxation may not drive all of the beam to strict thermodynamic equilibrium, we nevertheless assume it does as a working hypothesis. In view of the observations, we can expect this hypothesis to provide a realistic model of the interior regions of the beam and of the halo-generation mechanism, but a poorer model of the dynamics and evolution of the outermost regions of the halo.

We consider nonrelativistic beams for simplicity and because space-charge forces generally become less important to the dynamics as the beam energy grows. We select a plane orthogonal to the beam, superimpose a comoving coordinate system centered on the axis of the beam which coincides with the axis of the focusing channel, and describe the transverse dynamics in this coordinate system. Thus, our methodology can be straightforwardly applied to relativistic beams as well.

II. DYNAMICAL CONSIDERATIONS

A beam that is matched at injection into the focusing channel is stationary because it is in equilibrium and characterized by the Maxwell-Boltzmann distribution function [8]. If the beam is mismatched, it will evolve toward the Maxwell-Boltzmann distribution. Figure 1 is a flow chart depicting the very complicated evolutionary sequence which may ensue as a consequence of space charge. We constructed it based on the reported results of laboratory and numerical experiments.

A mismatched beam will undergo rapid charge redistribution during the first quarter of the beam's plasma period $t_p = 2\pi/w_p$ [13]. If the beam is rms matched, it becomes quasistationary at $t \simeq t_p/4$, at which time most of the free energy associated with mismatch of the shape of the density profile appears as emittance growth. The corresponding relaxation time t_r, which is the time needed for a typical particle to lose memory of its initial orbit, is of the order of the very long time scale t_b associated with binary Coulomb interactions, i.e., collisions in the absence of strong multiparticle correlations.

A rms-mismatched beam continues to oscillate. If the Debye length λ_D is much larger than the beam size a, then space charge is unimportant, the beam oscillates in response to the external focusing force, and the free energy associated with mismatch thermalizes on the time scale t_b so that $t_r \sim t_b$. If, on the other hand, λ_D is comparable to or smaller than a, then collective space-charge forces enter. Free energy associated with rms mismatch distributes itself in a spectrum of collective modes. These modes are subject to linear Landau (i.e., phase-mixed) damping. If the beam is Fourier transformed using the periodic boundary conditions of a homogeneous cube, a procedure which is strictly valid for a uniform and infinite configuration [14], then the time scale for linear Landau damping is found to be [10]

$$t_L(k) \simeq \frac{t_p}{2\pi}\sqrt{\frac{8}{\pi}}\,|k^3|\lambda_D^3\,\exp\left(\frac{1}{2k^2\lambda_D^2}+\frac{3}{2}\right) , \quad (2.1)$$

where k is the wave number of the mode. This equation is valid for $k\lambda_D \ll 1$. Modes for which $k\lambda_D \gtrsim 1$ damp on a time scale $\sim t_p$ and modes for which $k\lambda_D \ll 1$ are long

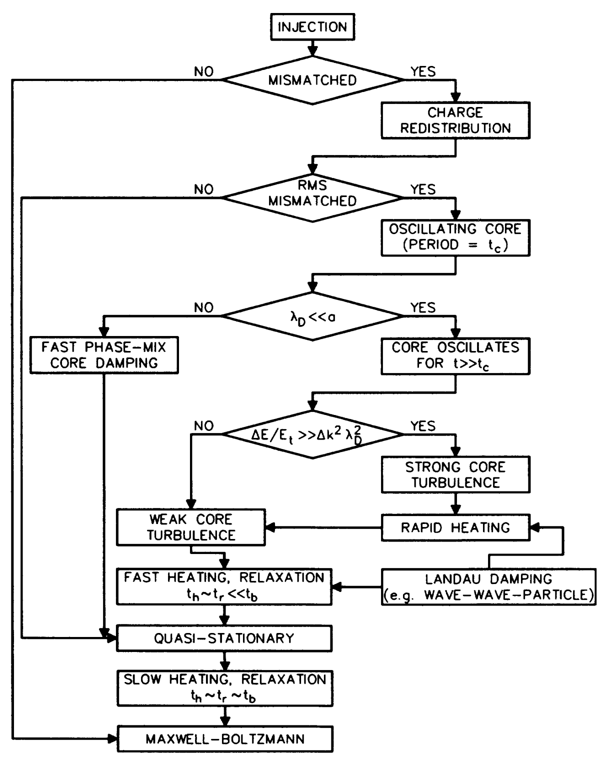

FIG. 1. Flowchart illustrating the evolutionary stages of mismatched charged-particle beams with space charge toward Maxwell-Boltzmann equilibrium.

lived. Thus, if $\lambda_D \gtrsim a$ (a warm beam), we can expect the modes to damp within a few core oscillations, but if $\lambda_D \ll a$ (a cool, space-charge-dominated beam), the large-scale global modes will persist.

There appear to be many degrees of freedom available by which a space-charge-dominated beam may release the free energy of rms mismatch. We may expect the beam to pass through a state of strong turbulence triggered by wave breaking associated with charge redistribution if

$$\Delta E_m/E_t \gg \Delta k^2 \lambda_D^2 , \tag{2.2}$$

where ΔE_m is the free energy per particle due to rms mismatch, E_t is the thermal energy per particle, and Δk is the effective half-width of the mode spectrum [15]. If the focusing channel is linear and continuous, we may express ΔE_m in the form [16]

$$\Delta E_m = \frac{1}{2} M v^2 k_0^2 a^2 h_m , \tag{2.3}$$

where M is the particle mass, v is the longitudinal velocity of the particle along the focusing channel, k_0 is the betatron wave number without space charge, a is the size of the equivalent uniform beam, and h_m is a dimensionless free-energy parameter

$$h_m = \frac{1}{2}\kappa^2(\rho_0^{-2}-1) - \frac{1}{2}(1-\rho_0^2) + (\kappa^2-1)\ln\rho_0 , \tag{2.4}$$

in which the tune depression κ is the ratio of the betatron wave number with space charge to k_0 and $\rho_0 = \tilde{x}_0/\tilde{x}_i$, where $\tilde{x}_0$ and $\tilde{x}_i$ are the rms beam sizes of the mismatched beam at $t = 0$ and the matched beam, respectively. Dividing by $E_t = Mv_t^2/2$, where v_t is the thermal velocity, gives

$$\frac{\Delta E_m}{E_t} = \left(\frac{v}{v_t} k_0 a\right)^2 h_m . \tag{2.5}$$

We may write the Debye length in terms of beam parameters by starting with

$$\frac{\lambda_D}{a} = \frac{v_t}{a\omega_p} . \tag{2.6}$$

The tune depression κ may be expressed as

$$\kappa^2 = 1 - \frac{K}{k_0^2 a^2} , \tag{2.7}$$

where the generalized perveance K incorporates the beam current I and magnitude of the particle charge Q in the manner

$$K = \frac{QI}{2\pi\varepsilon_0 M v^3} , \tag{2.8}$$

in which ε_0 is the permittivity of free space. Using $\omega_p^2 = nQ^2/\varepsilon_0 M$ and $I = \pi a^2 n Q v$, where n is the particle density, we have

$$\frac{\lambda_D}{a} = \frac{v_t}{v} \frac{1}{k_0 a\sqrt{2(1-\kappa^2)}} . \tag{2.9}$$

Let us now evaluate $\Delta E_m/E_t$ for examples of high-current proton and electron beams. Suppose $\rho_0 = 0.8$, $\kappa = 0.2$, $k_0 a = 0.2\pi$, and $v = 0.1c = 3\times 10^7$ m/s, for which $h_m = 0.045$. We choose temperatures T for the proton and electron beams which typify their respective sources. For a $T = 1$ eV proton beam, $v_t \sim 10^4$ m/s and $\Delta E_m/E_t \sim 10^5$. For a $T = 0.1$ eV electron beam, $v_t \sim 10^5$ m/s and $\Delta E_m/E_t \sim 10^3$. These beams are also space-charge dominated, for we have from Eq. (2.9) $\lambda_D/a \sim 10^{-4}$ for the proton beam and $\sim 10^{-3}$ for the electron beam. Since we expect $\Delta k \lambda_D$ to be of order unity or less, according to Eq. (2.2) these cold beams should pass through a state of strong turbulence in conjunction with wave breaking in phase space.

Localized collective modes in a strongly turbulent beam release their free energy very quickly (on a heating time scale $t_h \sim t_p$) via linear and/or nonlinear Landau damping. In so doing, the short-wavelength modes dissipate and single particles efficiently gain energy by repeated resonant interactions with the dissipating modes. The particles continue to interact with the residual weak turbulence and the beam relaxes on a time scale which is short compared to t_b [17]. For example, in three-dimensional beams which are strongly turbulent, the average "collision frequency" is $\sim g^{-1}$ times larger than in a quiescent beam, where $g \equiv 1/n\lambda_D^3$ is the plasma parameter [18]. In space-charge-dominated beams g^{-1} is large and interactions between particles and localized fluctuations are important. In weak turbulence the average collision frequency is $\sim g^{1/2}$ times smaller than in strong turbulence.

For example, in recent experiments with neutral plasmas the wave–wave-particle interaction is observed to generate heating. The particles extract energy from the waves as they proceed along dynamically chaotic orbits and the particles exhibit rapid stochastic diffusion in velocity space [19]. Furthermore, in conjunction with strong turbulence, long-wavelength modes can decompose into collapsing wave packets with length scales less than λ_D via the modulational, or ponderomotive, instability. This instability has been the subject of extensive analysis in conjunction with strong Langmuir turbulence in plasmas [15,20]. As their collapse progresses, the short-wavelength modes dissipate via Landau damping.

It is clear from these considerations that both systematic global oscillation of the potential and transient, stochastic local fluctuations can influence the violent relaxation of a beam. These processes increase the energy spread of the particles and they inject some particles into large-amplitude orbits, yet the ways they influence the particle distribution are qualitatively different.

The increase in energy spread, or "heating," reveals itself in the relative growth of the rms beam size $\tilde{x}$ and rms emittance $\tilde{\epsilon}$. Using the principle of conservation of energy, Reiser developed a recipe for estimating the final rms size $\tilde{x}_\infty$ and rms emittance $\tilde{\epsilon}_\infty$ from the free energy of mismatch [16] which we now summarize. Letting $\rho_\infty = \tilde{x}_\infty/\tilde{x}_i$, we have for a centered beam

$$\rho_\infty^2 - 1 - (1-\kappa^2)\ln\rho_\infty = h_s + h_m , \tag{2.10}$$

where the free energy h_s associated with shape mismatch is

$$h_s = \frac{1}{4}(1-\kappa^2)\nu , \tag{2.11}$$

in which ν is a dimensionless parameter (e.g., $\nu = 0.154$ for an initially Gaussian density profile) and h_m is given by Eq. (2.4). Equation (2.10) may be solved numerically for ρ_∞, from which we find

$$\frac{\tilde{x}_\infty}{\tilde{x}_0} = \frac{\rho_\infty}{\rho_0} \,. \tag{2.12}$$

The final emittance is given by

$$\frac{\tilde{\epsilon}_\infty}{\tilde{\epsilon}_0} = \frac{\tilde{x}_\infty}{\tilde{x}_0} \left[1 + \frac{1}{\kappa^2}(\rho_\infty^2 - 1)\right]^{1/2} \,. \tag{2.13}$$

These expressions lead to the identification of a final temperature. For example, suppose the injected beam is in thermodynamic equilibrium at temperature T_0 and the ejected beam is in thermodynamic equilibrium at temperature T_∞. Then the ratio of emittance to beam size is proportional to the rms velocity $\tilde{v} \propto \sqrt{T}$ at both $t = 0$ and $t \to \infty$, so $\tilde{\epsilon}_0/\tilde{x}_0 \propto \sqrt{T_0}$ and $\tilde{\epsilon}_\infty/\tilde{x}_\infty \propto \sqrt{T_\infty}$. Thus we identify

$$\frac{T_\infty}{T_0} = 1 + \frac{1}{\kappa^2}(\rho_\infty^2 - 1) \,. \tag{2.14}$$

If heating were absent, $\tilde{\epsilon}$ and $\tilde{x}$ would evolve together in lock-step, but when heating occurs, $\tilde{\epsilon}$ grows to a relatively larger amplitude than $\tilde{x}$, i.e., by the factor $(T_\infty/T_0)^{1/2}$. This circumstance typifies what is observed in laboratory and numerical experiments, and heating ($T_\infty > T_0$) is prevalent.

Calculations of the orbits of test particles in the electric potential of a globally oscillating core show that some particles gain energy, but the process is self-limiting [4, 11, 21, 22]. This is easily explained as a resonance phenomenon. A particle will gain energy if it enters the core when the core is large and leaves the core when the core is small. The space-charge force of the core then imparts a net boost to the particle and the process repeats itself as long as the particle orbits in phase with the core oscillations. However, as the particle increases its energy, its orbital period changes, the orbit eventually falls out of phase with the core oscillation, and the energy gain self-limits. There is accordingly a maximum amplitude which a particle can attain from this systematic process. For a given particle, the maximum amplitude clearly depends on its initial position and velocity. Inasmuch as the initial distribution function specifies the initial conditions of all particles, it also specifies the component of the halo governed by this process. In other words, *global oscillation of the potential does not generate additional halo.*

The time scale t_{cp} for ejection of particles via the oscillating-core–single-particle interaction can be estimated from elementary considerations. The change of total energy of a particle for each interaction with the core is [23]

$$\Delta E_p(\rho) \simeq \frac{1}{2} M v^2 k_0^2 a^2 (1-\kappa^2) F(\rho) \,, \tag{2.15}$$

where

$$F(\rho) = 4\frac{\rho+1}{\rho-1} - 2\left(\frac{\rho+1}{\rho-1}\right)^2 \ln\rho + 2\ln\rho \,, \tag{2.16}$$

and $\rho = \tilde{x}_{\rm in}/\tilde{x}_{\rm out}$, in which $\tilde{x}_{\rm in}$ and $\tilde{x}_{\rm out}$ are the rms beam sizes when the particle enters and leaves the core, respectively. For a particle which is resonant with the core oscillation, the maximum energy gain occurs when $\tilde{x}_{\rm in}$ is maximum and $\tilde{x}_{\rm out}$ is minimum, so we take $\rho = [\tilde{x}_0 + 2(\tilde{x}_i - \tilde{x}_0)]/\tilde{x}_0 = 2\rho_0^{-1} - 1$. The electric self-field of a uniform cylindrical beam is [24]

$$E(r) = \begin{cases} Q^{-1} M v^2 (1-\kappa^2) k_0 r & \text{for } r \le a \\ Q^{-1} M v^2 (1-\kappa^2) k_0 a^2/r & \text{for } r > a \,. \end{cases} \tag{2.17}$$

Upon superimposing the external focusing field, we find the single-particle potential energy to be

$$U(r) = \begin{cases} \frac{1}{2} M v^2 k_0^2 \kappa^2 r^2 \quad \text{for } r \le a \\ \frac{1}{2} M v^2 k_0^2 a^2 \Big\{ (\kappa^2 - 1)[1 + 2\ln(r/a)] + (r/a)^2 \Big\} \\ \qquad\qquad\qquad\qquad\qquad \text{for } r > a \,. \end{cases} \tag{2.18}$$

We define t_{cp} to be the time it takes for the resonant particle, which gains energy $\Delta E_p(\rho)$ once each period t_c of core oscillation, to climb up the potential well from $U(r = a)$ to $U(r = 2a)$:

$$t_{cp} \sim t_c \frac{U(2a) - U(a)}{\Delta E_p(2\rho_0^{-1} - 1)} = t_c \frac{3 - (1-\kappa^2) 2\ln 2}{(1-\kappa^2) F(2\rho_0^{-1} - 1)} \,. \tag{2.19}$$

For $\kappa = 0.2$ and $\rho_0 = 0.8$, we have $t_{cp} \simeq 3t_c$, which is representative. Numerical experiments with space-charge-dominated beams typically show particles appearing well outside the core after just one or two core oscillations.

By contrast, interactions with a broad spectrum of modes are not self-limiting. A statistically small sample of particles may be expected to interact in phase with many localized fluctuations over several orbital periods, and the orbits of these particles would thereby achieve very large amplitudes. Moreover, these interactions will divert some particles from orbits which are nonresonant with respect to the core oscillation to resonant orbits and these particles will then experience the self-limiting energy gain. Inasmuch as these mode-particle interactions are stochastic, the attendant dynamics is very difficult to analyze, but it is clear that particles can quickly lose memory of their initial conditions. *Localized fluctuations thereby provide the mechanism for generating additional halo.* There is both experimental and numerical evidence for ongoing population of the halo by particles having small initial energy [3,4,11].

Macroscopically, interactions of the particles with turbulent fluctuations produce rapid heating at the expense of the available free energy and at a rate commensurate with the strength of the turbulence. Therefore, from a thermodynamic viewpoint, we regard the initial particle distribution function and its associated temperature T_0, and the final (Maxwell-Boltzmann) particle distribution function and its associated temperature T_∞, to be given. Our goal now is to develop a formalism to model the dynamics connecting the initial distribution function to the final distribution function.

III. FOKKER-PLANCK MODEL

A precise statistical treatment of the beam dynamics involves the microscopic Klimontovich density distribution. This distribution consists of a self-consistent superposition of the orbits of all constituent particles of the beam in the Cartesian phase space $(\mathbf{x}, \mathbf{u})$ of a single particle and it satisfies Liouville's theorem [25, 26]. The microscopic distribution is

$$f(\mathbf{x}, \mathbf{u}, t) = \frac{1}{N} \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{x}_i(t)) \delta(\mathbf{u} - \mathbf{u}_i(t)) , \quad (3.1)$$

where $(\mathbf{x}_i(t), \mathbf{u}_i(t))$ denotes the orbit of the ith particle and N is the total number of particles. The ultimate goal of a numerical experiment is to specify the Klimontovich distribution as a function of time.

To develop an analytic formalism, it is more practical to work with a macroscopic distribution function. We thus consider the macroscopic, coarse-grained distribution function $\bar{f}(\mathbf{x}, \mathbf{u}, t)$, which is found by averaging the Klimontovich distribution over scales substantially greater than those associated with localized turbulent fluctuations:

$$\bar{f}(\mathbf{x}, \mathbf{u}, t) = \frac{1}{\Delta V(\mathbf{x}, \mathbf{u})} \int_{\Delta V(\mathbf{x}, \mathbf{u})} d\mathbf{x}' \, d\mathbf{u}' \, f(\mathbf{x}', \mathbf{u}', t) , \quad (3.2)$$

where $\Delta V(\mathbf{x}, \mathbf{u})$ is a phase-space volume element centered on the coordinates $(\mathbf{x}, \mathbf{u})$ which is large compared to the size of the turbulent fluctuations but small compared to the size of the beam. In what follows, the overbar is a signature of quantities which are calculated from the coarse-grained distribution $\bar{f}$.

After resolving the Klimontovich distribution into two components, $\bar{f}$ and fluctuations about $\bar{f}$, and averaging Liouville's equation, we are left with a "collision" term involving the fluctuations [26]. Working with a coarse-grained distribution function is tantamount to neglecting nonlinear coupling between fluctuations in the particle distribution and fluctuations in the electromagnetic field. This approach results in the reduction of Liouville's equation to an equation of the Fokker-Planck type [17, 18, 27]:

$$(\partial_t + \mathbf{u} \cdot \nabla_x + \overline{\mathbf{K}} \cdot \nabla_u) \bar{f} = \nabla_u \cdot (\mathbf{F} \bar{f}) + \nabla_u \cdot (\mathsf{D} \cdot \nabla_u \bar{f}) , \quad (3.3)$$

where $\overline{\mathbf{K}}$ is the net acceleration of a particle in the comoving frame found from the potentials Φ_f and $\bar{\Phi}_s$ associated with the external focusing force and coarse-grained internal space-charge force, respectively, i.e.,

$$\overline{\mathbf{K}} = -QM^{-1} \nabla_x (\Phi_f + \bar{\Phi}_s) , \quad (3.4)$$

and the vector $\mathbf{F}$ and tensor D are coefficients of friction and diffusion, respectively. According to Poisson's equation, $\bar{\Phi}_s$ is determined from the coarse-grained density, which is in turn determined from $\bar{f}$:

$$\nabla_x^2 \bar{\Phi}_s(\mathbf{x}, t) = -\frac{NQ}{\varepsilon_0} \int d\mathbf{u} \, \bar{f}(\mathbf{x}, \mathbf{u}, t) . \quad (3.5)$$

The right-hand side of Eq. (3.3) drives the final velocity distribution to the Maxwellian distribution, in which the velocity of the center of charge of the system is constant and the coordinate system is comoving with the center of charge. For a beam in a focusing channel this implies a centered beam with $\langle \mathbf{x} \rangle = \mathbf{0}$ and $\langle \mathbf{u} \rangle = \mathbf{0}$. Although a misaligned beam may migrate to the static equilibrium of a centered beam, the time scale for the decay of the beam centroid's betatron oscillation may be different from the relaxation time. For example, the oscillation persists if the external focusing force is linear [28]. Thus we restrict this analysis to beams which are initially centered.

If the coarse-grained beam is regarded to be uniform so that it can be Fourier transformed using the periodic boundary conditions of a homogeneous cube of volume V, then the transport coefficients are [14, 18]

$$\mathbf{F} = \frac{1}{V} \frac{2\pi Q^2}{M\varepsilon_0} \sum_{\mathbf{k}} \frac{\mathbf{k}}{k^2} \frac{\delta(\omega_{\mathbf{k}} - \mathbf{k} \cdot \mathbf{u})}{\epsilon'(\mathbf{k}, \omega_{\mathbf{k}})} \quad (3.6)$$

and

$$\mathsf{D} = \frac{1}{V} \frac{2\pi Q^2}{M^2 \varepsilon_0} \sum_{\mathbf{k}} \frac{\mathcal{E}_{\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{\mathbf{k}\mathbf{k}}{k^2} \frac{\delta(\omega_{\mathbf{k}} - \mathbf{k} \cdot \mathbf{u})}{\epsilon'(\mathbf{k}, \omega_{\mathbf{k}})} . \quad (3.7)$$

In these expressions, $\mathcal{E}_{\mathbf{k}}$ is the energy contained in the fluctuation with wave vector $\mathbf{k}$ and angular frequency $\omega_{\mathbf{k}}$, and in this quasilinear formulation it evolves in the manner

$$\frac{\partial \mathcal{E}_{\mathbf{k}}}{\partial t} = 2\gamma_{\mathbf{k}} \mathcal{E}_{\mathbf{k}} + \frac{\pi M \omega_p^2 \omega_{\mathbf{k}}}{k^2 \epsilon'(\mathbf{k}, \omega_{\mathbf{k}})} \int d\mathbf{u} \, \bar{f}(\mathbf{u}) \delta(\omega_{\mathbf{k}} - \mathbf{k} \cdot \mathbf{u}) , \quad (3.8)$$

where

$$\gamma_{\mathbf{k}} \simeq \left[\frac{\partial \mathrm{Re}\,[\epsilon(\mathbf{k}, \omega)]}{\partial \omega} \right]^{-1} \mathrm{Im}\,[\epsilon(\mathbf{k}, \omega_{\mathbf{k}})] . \quad (3.9)$$

The first term on the right-hand side of Eq. (3.8) accounts for Landau damping or growth at the rate $\gamma_{\mathbf{k}}$ from absorption or induced emission of mode energy by the particles, respectively, and the second term accounts for spontaneous Cherenkov emission of mode energy by the particles. The dielectric response function is

$$\epsilon(\mathbf{k}, \omega) = 1 + \frac{\omega_p^2}{k^2} \int d\mathbf{u} \frac{\mathbf{k} \cdot \nabla_u \bar{f}(\mathbf{u})}{\omega - \mathbf{k} \cdot \mathbf{u}} . \quad (3.10)$$

$\epsilon'(\mathbf{k}, \omega_{\mathbf{k}})$ denotes $[\partial \epsilon / \partial \omega]_{\omega = \omega_{\mathbf{k}}}$, $\omega = \omega_{\mathbf{k}} + i\gamma_{\mathbf{k}}$ is the solution of $\epsilon(\mathbf{k}, \omega) = 0$, and the $\mathbf{k}$ summation in Eqs. (3.6) and (3.7) is over the growing (unstable) modes obtained from these zeros, for which $\gamma_{\mathbf{k}} > 0$. The friction $\mathbf{F}$ arises from particles losing energy to unstable modes via spontaneous Cherenkov emission and the diffusion D results from particles recoiling in response to absorption and induced emission. In turn, diffusion gives rise to turbulent heating of the particles.

The effect of fluctuations is to change the shape of $\bar{f}$ continuously until there are no more growing modes and $\gamma_{\mathbf{k}} < 0$ for all $\mathbf{k}$. When this has occurred, the modes quickly dissipate by linear Landau damping and the turbulence vanishes. For example, in the presence of a background isotropic Maxwellian velocity distribution, the fluctuation spectrum evolves as

$$\frac{\partial \mathcal{E}_{\mathbf{k}}}{\partial t} = \frac{\pi M \omega_p^2 \omega_{\mathbf{k}}}{k^2 \epsilon'(\mathbf{k}, \omega_{\mathbf{k}})} \left(1 - \frac{\mathcal{E}_{\mathbf{k}}}{k_B T}\right) \int d\mathbf{u}\, \bar{f}(\mathbf{u}) \delta(\omega_{\mathbf{k}} - \mathbf{k} \cdot \mathbf{u}) , \tag{3.11}$$

in which k_B is Boltzmann's constant. This shows explicitly that $\mathcal{E}_{\mathbf{k}}$ dissipates if $\mathcal{E}_{\mathbf{k}} > k_B T$ and the energy spectrum thermalizes toward the equipartition value $\mathcal{E}_{\mathbf{k}} = k_B T$.

We have collected this quasilinear formalism from Ichimaru's textbooks [17, 18] to highlight the ingredients of a self-consistent solution for the dynamics of a turbulent beam. As the fluctuations evolve, they change the shape of the coarse-grained distribution function, which in turn modifies the evolution of the fluctuations. The formalism is obviously very complicated. It is also only approximate, particularly with regard to its application to an inhomogeneous beam, for in this case it is very difficult to calculate the normal modes. In the spirit of our goal to develop a semianalytic formalism, we shall replace the complicated theory with a simple model of the turbulence which preserves the salient features of a fully self-consistent approach.

In general, the transport coefficients may be expected to be functions of position, velocity, and time. We shall ignore the position and velocity dependences and model the beam as a fluctuating fluid in which particles execute Brownian motion as they interact with the fluctuations. We take $\mathbf{F} = \beta(t)\mathbf{u}$ and $\mathsf{D} = D(t)\mathsf{I}$, where $\beta(t)$ and $D(t)$ are time-dependent relaxation-rate and diffusion coefficients, respectively, and I is the identity tensor. These simplifications lead to our model Fokker-Planck equation

$$(\partial_t + \mathbf{u} \cdot \nabla_x + \overline{\mathbf{K}} \cdot \nabla_u)\bar{f} = \beta \nabla_u \cdot (\mathbf{u}\bar{f}) + D\nabla_u^2 \bar{f} . \tag{3.12}$$

The left-hand side accounts for systematic effects arising from the external focusing field and the mean space-charge field of the entire beam, and it therefore includes resonances between global space-charge modes and the focusing force if any are present. It also includes the systematic dynamics of the oscillating-core–single-particle interaction described in Sec. II above. The right-hand side accounts for stochastic effects of the collective-mode spectrum, specifically the localized fluctuations, on the particle orbits.

Equation (3.12) will be most accurate for slow-moving particles confined to the central region of the beam. Faster particles will have less time to interact with the fluctuations and they will sample the outer region of the beam in which the density profile has a strong spatial dependence. Thus, by ignoring the position and velocity dependencies of $\mathbf{F}$ and D and forcing D to be isotropic, we are accounting principally for the dynamics of the bulk of the particles confined to the inner part of the beam and for the expulsion of particles from the core and less accurately for the evolution of particles comprising the outermost regions of the halo. This is the gist of the philosophy set forth in the Introduction.

IV. ONE-DIMENSIONAL BEAMS

In the case of one-dimensional (1D) beams we assume that the coarse-grained distribution $\bar{f}(\mathbf{x}, \mathbf{u}, t)$ is independent of the longitudinal position z and of the transverse position y. We also assume that the dependences on the velocities u_z and u_y are separable and that we have already integrated $\bar{f}$ over u_z and u_y. We let $\overline{W}(x, u, t)$ represent the integrated distribution function. It then satisfies the Fokker-Planck equation

$$(\partial_t + u\partial_x + \overline{K}\partial_u)\overline{W} = \beta\partial_u(u\overline{W}) + D\partial_u^2\overline{W} , \tag{4.1}$$

where $\overline{K} = K_f + \overline{K}_s$ is the total acceleration experienced by a particle. The distribution $\overline{W}$ is normalized such that

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} du\, \overline{W}(x, u, t) = 1 . \tag{4.2}$$

The coarse-grained density $\bar{n}(x, t)$ is obtained from the distribution $\overline{W}(x, u, t)$:

$$\bar{n}(x, t) = L^{-2} \int_{-\infty}^{+\infty} du\, \overline{W}(x, u, t) , \tag{4.3}$$

where L^{-2} is the total number of particles per unit area in the (y, z) plane. We shall consider a continuous, linear focusing channel for which $K_f = -\omega^2 x$. The acceleration $\overline{K}_s$ due to space-charge forces is determined from the density $\bar{n}(x, t)$ by Poisson's equation

$$\partial_x \overline{K}_s = \frac{Q^2}{M\varepsilon_0}\, \bar{n}(x, t) . \tag{4.4}$$

Since β and D are taken to be independent of x and u, the right-hand side of the Fokker-Planck equation can be written as $\beta\mathcal{L}_u[\overline{W}]$, where

$$\mathcal{L}_u = \partial_u[u + (2\alpha)^{-1}\partial_u] \tag{4.5}$$

is the Fokker-Planck collision operator in which $\alpha(t) = \beta/2D$. The Gauss-Hermite functions $\psi_m(u)$, defined as

$$\psi_m(u) = \left(\frac{1}{2^m m!}\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha u^2} H_m(\sqrt{\alpha}u) , \tag{4.6}$$

are eigenfunctions of $\mathcal{L}_u$, i.e., $\mathcal{L}_u[\psi_m] = -m\psi_m$. We therefore use the following expansion for the velocity dependence of the distribution function:

$$\overline{W}(x, u, t) = \sum_{m=0}^{\infty} A_m(x, t)\, \psi_m(u) . \tag{4.7}$$

We then expand the coefficients $A_m(x, t)$ in terms of the Gauss-Hermite functions $\varphi_n(x)$:

$$\varphi_n(x) = \left(\frac{1}{2^n n!}\frac{a}{\pi}\right)^{1/2} e^{-ax^2} H_n(\sqrt{a}x) , \tag{4.8}$$

in which $a(t)$ is a free time-dependent variable. The full decomposition for the distribution function is then

$$\overline{W}(x, u, t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_m^n(t)\, \psi_m(u)\, \varphi_n(x) . \tag{4.9}$$

This decomposition has the advantage that its leading term is Gaussian both in velocity and position. The leading term therefore corresponds to the Maxwell-Boltzmann equilibrium distribution appropriate to the continuous, linear focusing channel in the absence of space charge and it thus provides a solid foundation on which to build the effects of nonlinear space-charge forces. Another advantage is that the conservation of the total number of particles is expressed simply as $A_0^0(t) = 1$ for all t. This is obtained directly from the particle density

$$\bar{n}(x,t) = L^{-2} \sum_{n=0}^{\infty} A_0^n(t)\, \varphi_n(x) \,, \tag{4.10}$$

which, when integrated over x, must always yield L^{-2}, and from the properties of the Gauss-Hermite functions which are summarized in Appendix A. Additionally, as shown later in this section, the requirement that the beam is centered implies that $A_1^0(t) = A_0^1(t) = 0$ for all t.

The acceleration $\overline{K}_s$ is found by straightforward integration to be

$$\overline{K}_s(x,t) = \frac{Q^2}{2LM\varepsilon_0}\left[A_0^0(t)\,\mathrm{erf}(\sqrt{a}x) - \sqrt{\frac{2}{a}} \sum_{j'=1}^{\infty} \frac{A_0^{j'}(t)}{\sqrt{j'}}\, \varphi_{j'-1}(x)\right] . \tag{4.11}$$

As shown in Appendix B, the Fokker-Planck equation yields the following set of coupled nonlinear differential equations for the coefficients A_m^n:

$$d_\zeta A_m^n = \frac{d_\zeta \hat{\alpha}}{2\hat{\alpha}}\left[mA_m^n + \sqrt{m(m-1)}A_{m-2}^n\right] + \frac{d_\zeta \hat{a}}{2\hat{a}}\left[nA_m^n + \sqrt{n(n-1)}A_m^{n-2}\right] + \sqrt{\frac{\hat{a}}{\hat{\alpha}}}\sqrt{n}\left[\sqrt{m}A_{m-1}^{n-1} + \sqrt{m+1}A_{m+1}^{n-1}\right]$$
$$- \sqrt{\frac{\hat{\alpha}}{\hat{a}}}\frac{\omega^2}{\omega_0^2}\sqrt{m}\left[\sqrt{n}A_{m-1}^{n-1} + \sqrt{n+1}A_{m-1}^{n+1}\right] - \frac{\omega_p^2}{\omega_0^2}\sqrt{\hat{\alpha}}\sqrt{m}\sum_{j=0}^{\infty} A_{m-1}^j \sum_{k=0}^{\infty} A_0^k\, I_{jn}^k - m\frac{\beta}{\omega_0}A_m^n \,, \tag{4.12}$$

in which $\omega_0 = \sqrt{a(0)/\alpha(0)}$ is a reference frequency, $\zeta = \omega_0 t$ is the dimensionless time, $\hat{\alpha} = \alpha(\zeta)/\alpha(0)$, $\hat{a} = a(\zeta)/a(0)$, and

$$\omega_p = \sqrt{\frac{Q^2}{ML^2\varepsilon_0}\sqrt{\frac{a(0)}{2}}} \tag{4.13}$$

is a plasma frequency. The coefficients I_{jn}^k are defined in Appendix B.

It is apparent upon inspection of the system (4.12) that the space-charge force introduces a quadratic coupling between the coefficients. In the absence of space charge ($\omega_p = 0$), the system (4.12) becomes a set of coupled linear differential equations. This suggests that an analytical solution is possible for this case, and it is presented in Sec. VI. The last term in Eq. (4.12) is a damping term and will govern the behavior for $t \to \infty$. Since it is proportional to m, only the coefficients A_0^n will maintain a finite value when $t \to \infty$. In the case of a beam which is symmetric with respect to the axis at all times, only the coefficients with $m+n$ even are needed since $A_m^n = 0$ if $m+n$ is odd.

The center position of the beam $\langle x \rangle$, the mean transverse velocity $\langle u \rangle$, the mean-square width $\langle x^2 \rangle$, the mean-square velocity $\langle u^2 \rangle$, and the mean-square emittance $\epsilon^2 = \langle x^2 \rangle \langle u^2 \rangle - \langle xu \rangle^2$ can be expressed simply in terms of the expansion coefficients

$$\langle x \rangle = A_0^1/a^{1/2} = 0 \quad \text{by assumption,} \tag{4.14}$$

$$\langle u \rangle = A_1^0/\alpha^{1/2} = 0 \quad \text{by assumption,} \tag{4.15}$$

$$\langle x^2 \rangle = \frac{1}{2a}\left(1 + \sqrt{2}A_0^2\right) , \tag{4.16}$$

$$\langle u^2 \rangle = \frac{1}{2\alpha}\left(1 + \sqrt{2}A_2^0\right) , \tag{4.17}$$

$$\epsilon^2 = \frac{1}{4a\alpha}\left[\left(1 + \sqrt{2}A_0^2\right)\left(1 + \sqrt{2}A_2^0\right) - \left(A_1^1\right)^2\right] . \tag{4.18}$$

The parameter $a(\zeta)$ is still arbitrary. This follows from the fact that the decomposition of the distribution function in physical space is not unique. One possible choice is to assume that $a(\zeta)$ has the same time dependence as $\alpha(\zeta)$ as was done in our earlier paper [12]. Another possibility is to use the close connection between $a(\zeta)$ and the mean-square width of the beam as indicated by Eq. (4.16). Imposing the condition $A_0^2(\zeta) \equiv 0$ will yield $\langle x^2(\zeta) \rangle = [2a(\zeta)]^{-1}$. This condition will be satisfied if $A_0^2(0) = 0$ and $d_\zeta A_0^2 = 0$. The latter will be obtained if we require $\hat{a}$ to satisfy

$$d_\zeta \hat{a} = -2\hat{a}\sqrt{\frac{\hat{a}}{\hat{\alpha}}}A_1^1 \,. \tag{4.19}$$

Specifying the parameter $a(t)$ [or equivalently the parameter $\hat{a}(\zeta)$] by imposing condition (4.19) identifies $a(t)$ with the reciprocal of the instantaneous mean-square width. In particular it guarantees that, in the absence of space charge and when equilibrium is reached ($t \to \infty$), all the coefficients A_m^n are zero except $A_0^0 = 1$.

V. TWO-DIMENSIONAL BEAMS

In the case of two-dimensional (2D) beams, we assume that the coarse-grained distribution $\bar{f}(\mathbf{x},\mathbf{u},t)$ is independent of the longitudinal position z, that the dependence on the longitudinal velocity u_z is separable, and that the distribution function has been integrated over u_z. We represent the integrated distribution in polar coordinates by $\overline{W}(r,\theta,u_r,u_\theta,t)$, which satisfies the Fokker-Planck equation

$$\left[\partial_t + u_r\,\partial_r + \frac{u_\theta}{r}\,\partial_\theta + \left(\overline{K}_r + \frac{u_\theta^2}{r}\right)\partial_{u_r} + \left(\overline{K}_\theta - \frac{u_r u_\theta}{r}\right)\partial_{u_\theta}\right]\overline{W} = \beta\,(\mathcal{L}_{u_r} + \mathcal{L}_{u_\theta})\,\overline{W}\,, \quad (5.1)$$

where $\mathcal{L}_{u_r}$ and $\mathcal{L}_{u_\theta}$ are the Fokker-Planck collision operators defined in Eq. (4.5) for the radial and azimuthal velocities, respectively. The distribution $\overline{W}$ is normalized such that

$$\int_0^\infty dr\, r\int_0^{2\pi} d\theta\int_{-\infty}^{+\infty} du_r \int_{-\infty}^{+\infty} du_\theta\, \overline{W}(r,\theta,u_r,u_\theta,t) = 1\,. \quad (5.2)$$

We shall consider a nonrotating, generally nonaxisymmetric, centered beam in a continuous, linear focusing channel for which $\overline{K}_r = -\omega^2 r$. The acceleration $\overline{K}_s$ due to space-charge forces is determined from the density $\bar{n}(r,\theta,t)$ by Poisson's equation

$$\nabla_x\cdot\overline{\mathbf{K}}_s = \frac{Q^2}{M\varepsilon_0}\bar{n}(r,\theta,t)\,. \quad (5.3)$$

To solve the Fokker-Planck and Poisson equations self-consistently, we expand the distribution function into complete sets of orthogonal functions:

$$\overline{W}(r,\theta,u_r,u_\theta,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=-\infty}^{+\infty}\sum_{q=0}^{\infty} A^{p,q}_{m,n}(t)\,\psi_m(u_r) \times\psi_n(u_\theta)\,\phi^p_q(r)\,e^{ip\theta}\,. \quad (5.4)$$

Here $\psi_m(u_r)$ and $\psi_n(u_\theta)$ are the Gauss-Hermite functions defined in Eq. (4.6) and $\phi^p_q(r)$ are the Gauss-Laguerre functions

$$\phi^p_q(r) = \frac{a}{\pi}\left[\frac{q!}{(|p|+q)!}\right]^{1/2}(ar^2)^{|p|/2}\,e^{-ar^2}\,L^{|p|}_q(ar^2) \quad (5.5)$$

in which $a(t)$ is again a free time-dependent variable. The leading term in this expansion is Gaussian in u_r, u_θ, and r, i.e., it is the Maxwell-Boltzmann equilibrium distribution in the absence of space charge appropriate to a continuous, linear focusing channel. The coefficients $A^{p,q}_{m,n}$ are complex and satisfy $A^{p,q}_{m.n} = (A^{-p,q}_{m,n})^*$ and they can also be expressed in terms of their real and imaginary parts

$$C^{p,q}_{m,n} = \mathrm{Re}(A^{p,q}_{m,n}), \qquad S^{p,q}_{m,n} = \mathrm{Im}(A^{p,q}_{m,n})\,. \quad (5.6)$$

The coarse-grained density $\bar{n}(r,\theta,t)$ is

$$\bar{n}(r,\theta,t) = L^{-1}\int_{-\infty}^{+\infty} du_r\int_{-\infty}^{+\infty} du_\theta\,\overline{W}(r,\theta,u_r,u_\theta,t) = L^{-1}\sum_{p=-\infty}^{+\infty}\sum_{q=0}^{\infty} A^{p,q}_{0,0}(t)\,\phi^p_q(r)\,e^{ip\theta}\,, \quad (5.7)$$

where L^{-1} is the total number of particles per unit length in the z direction, which must be obtained by integrating $\bar{n}(r,\theta,t)$ over r and θ. Using the properties of the Gauss-Laguerre functions, this implies $A^{0,0}_{0,0}(t) = 1$ for all t. There are additional conditions on the coefficients which are imposed by the properties that the beam is centered and nonrotating. These conditions will be presented later in this section.

The components $\overline{K}_r$ and $\overline{K}_\theta$ of the acceleration are found by solving Poisson's equation, which is done in Appendix C. They are

$$\overline{K}_r = -\omega^2 r + \frac{\omega_p^2}{a_0 r}\Bigg\{(1-e^{-ar^2}) - \sum_{\substack{p=-\infty\\p\neq0}}^{+\infty} A^{p,0}_{0,0}\left[\frac{(ar^2)^{|p|}}{|p|!}\right]^{1/2}\left[e^{-ar^2} - \frac{|p|}{2}(ar^2)^{-|p|}\gamma(|p|,ar^2)\right]e^{ip\theta} - \frac{\pi}{a}\sum_{p=-\infty}^{+\infty}\sum_{q=1}^{\infty} A^{p,q}_{0,0}\left[\phi^p_q(r) - \frac{2q+|p|}{2\sqrt{q(|p|+q)}}\,\phi^p_{q-1}(r)\right]e^{ip\theta}\Bigg\}\,, \quad (5.8)$$

$$\overline{K}_\theta = -\frac{\omega_p^2}{2a_0 r}\sum_{\substack{p=-\infty\\p\neq0}}^{+\infty}\left\{A^{p,0}_{0,0}\left[\frac{(ar^2)^{-|p|}}{|p|!}\right]^{1/2}\gamma(|p|,ar^2) + \frac{\pi}{a}\sum_{q=1}^{\infty} A^{p,q}_{0,0}\frac{1}{\sqrt{q(|p|+q)}}\phi^p_{q-1}\right\}ip\,e^{ip\theta}\,, \quad (5.9)$$

in which we introduce a plasma frequency for the two-dimensional beam

$$\omega_p = \sqrt{\frac{Q^2 a_0}{2\pi L M\varepsilon_0}} \quad (5.10)$$

and $\gamma(p,x)$ is the incomplete gamma function.

As shown in Appendix C, the Fokker-Planck equation yields the following set of coupled nonlinear differential equations for the coefficients $A^{p,q}_{m,n}$:

$$
\begin{aligned}
d_\zeta A^{p,q}_{m,n} = {} & \frac{d_\zeta \hat{\alpha}}{2\hat{\alpha}} \left[\sqrt{m(m-1)}A^{p,q}_{m-2,n} + (m+n)A^{p,q}_{m,n} + \sqrt{n(n-1)}A^{p,q}_{m,n-2}\right] + \frac{d_\zeta \hat{a}}{2\hat{a}} \left[(|p|+2q)A^{p,q}_{m,n} - 2\sqrt{q(|p|+q)}A^{p,q-1}_{m,n}\right] \\
& + \sqrt{\frac{\hat{a}}{2\hat{\alpha}}}\sqrt{m}\sum_{q'=0}^{\infty} A^{p,q'}_{m-1,n}\left\{\left[|p|+2(q'+1)+n\right]J^{p}_{q\,q'} - 2\sqrt{(q'+1)(q'+|p|+1)}J^{p}_{q\,q'+1}\right\} \\
& + \sqrt{\frac{\hat{a}}{2\hat{\alpha}}}\sqrt{m+1}\sum_{q'=0}^{\infty} A^{p,q'}_{m+1,n}\left\{\left[|p|+2(q'+1)-(n+1)\right]J^{p}_{q\,q'} - 2\sqrt{(q'+1)(q'+|p|+1)}J^{p}_{q\,q'+1}\right\} \\
& + \sqrt{\frac{\hat{a}}{2\hat{\alpha}}}\sum_{q'=0}^{\infty} J^{p}_{q\,q'}\left\{\sqrt{m(n+1)(n+2)}A^{p,q'}_{m-1,n+2} - \sqrt{n(n-1)(m+1)}A^{p,q'}_{m+1,n-2}\right. \\
& \left. - ip\left(\sqrt{n}A^{p,q'}_{m,n-1} + \sqrt{n+1}A^{p,q'}_{m,n+1}\right)\right\} - \sqrt{\frac{2\hat{\alpha}}{\hat{a}}}\frac{\omega^2}{\omega_0^2}\sqrt{m}\sum_{q'=0}^{\infty} A^{p,q'}_{m-1,n}K^{p}_{q\,q'} \\
& - \sqrt{\frac{\hat{a}}{2\hat{\alpha}}}\hat{\alpha}\frac{\omega_p^2}{\omega_0^2}\sum_{p'=-\infty}^{+\infty}\sum_{q'=0}^{\infty}\sum_{p''=-\infty}^{+\infty}\sum_{q''=0}^{\infty} A^{p'',q''}_{0,0}\,\delta(p-p'-p'')\left(2\sqrt{m}A^{p',q'}_{m-1,n}M^{p\,p'\,p''}_{q\,q'\,q''} - i\sqrt{n}A^{p',q'}_{m,n-1}N^{p\,p'\,p''}_{q\,q'\,q''}\right) \\
& - (m+n)\frac{\beta}{\omega_0}A^{p,q}_{m,n}\ ,
\end{aligned}
\tag{5.11}
$$

where $\hat{a}$, $\hat{\alpha}$, ω_0, and ζ have the same definitions as in Sec. IV, $\delta(p)$ is the Kronecker delta function, and the constant coefficients $J^{p}_{q\,q'}$, $K^{p}_{q\,q'}$, $M^{p\,p'\,p''}_{q\,q'\,q''}$, and $N^{p\,p'\,p''}_{q\,q'\,q''}$ are defined in Appendix C. The last term in Eq. (5.11) acts as a damping term and will govern the behavior as $t \to \infty$. Since it is proportional to $(m+n)$, only the coefficients $A^{p,q}_{0,0}$ will maintain a finite value when $t \to \infty$. This results from the requirement that the velocity distribution in u_r and u_θ evolve toward a Maxwellian distribution.

It is apparent upon inspection of the system (5.11) that the space-charge force introduces a quadratic coupling between the coefficients. In the absence of space charge ($\omega_p = 0$), the system (5.11) becomes a set of coupled linear differential equations. This suggests that an analytical solution is possible for this case and it is presented in Sec. VI for an axisymmetric beam.

From the coefficients $A^{p,q}_{m,n}$, or their real and imaginary parts $C^{p,q}_{m,n}$ and $S^{p,q}_{m,n}$, respectively, various moments can be calculated:

$$\langle x\rangle = a^{-1/2}C^{1,0}_{0,0} = 0 \quad \text{by assumption}\ , \tag{5.12}$$

$$\langle y\rangle = a^{-1/2}S^{1,0}_{0,0} = 0 \quad \text{by assumption}\ , \tag{5.13}$$

$$\langle u_x\rangle = \frac{1}{2\sqrt{2\alpha}}\sum_{q=0}^{\infty}\frac{\Gamma(q+\frac{1}{2})}{q!\sqrt{q+1}}(C^{1,0}_{1,0} + S^{1,0}_{0,1}) = 0 \quad \text{by assumption}\ , \tag{5.14}$$

$$\langle u_y\rangle = \frac{1}{2\sqrt{2\alpha}}\sum_{q=0}^{\infty}\frac{\Gamma(q+\frac{1}{2})}{q!\sqrt{q+1}}(C^{1,0}_{0,1} + S^{1,0}_{1,0}) = 0 \quad \text{by assumption}, \tag{5.15}$$

$$\langle u_\theta\rangle = (2\alpha)^{-1/2}C^{0,0}_{0,1} = 0 \quad \text{by assumption}\ , \tag{5.16}$$

$$\langle u_r\rangle = (2\alpha)^{-1/2}C^{0,0}_{1,0}\ , \tag{5.17}$$

$$\langle r^2\rangle = a^{-1}\left(1 - C^{0,1}_{0,0}\right)\ . \tag{5.18}$$

Equations (5.12)–(5.15) imply that the centering of the beam is governed by the absence of dipole terms ($p = \pm 1$). A generalized emittance, which is appropriate for use with beams in which the Cartesian components of the motion are coupled, can also be calculated [29]:

$$
\begin{aligned}
\epsilon_2^2 = {} & \langle x^2\rangle\langle u_x^2\rangle - \langle xu_x\rangle^2 + \langle y^2\rangle\langle u_y^2\rangle - \langle yu_y\rangle^2 \\
& + 2\langle xy\rangle\langle u_xu_y\rangle - 2\langle xu_y\rangle\langle yu_x\rangle .
\end{aligned}
\tag{5.19}
$$

This generalized emittance takes the form

$$
\begin{aligned}
64a\alpha\epsilon_2^2 = {} & 16\left(1 - C^{0,1}_{0,0}\right)\left[2 + \sqrt{2}\left(C^{0,0}_{2,0} + C^{0,0}_{0,2}\right)\right] + 32\,C^{2,0}_{0,0}\sum_{q=0}^{\infty}\frac{\left(C^{2,q}_{2,0} - C^{2,q}_{0,2}\right) + \sqrt{2}S^{2,q}_{1,1}}{\sqrt{(q+1)(q+2)}} \\
& + 32\,S^{2,0}_{0,0}\sum_{q=0}^{\infty}\frac{\left(S^{2,q}_{2,0} - S^{2,q}_{0,2}\right) + \sqrt{2}C^{2,q}_{1,1}}{\sqrt{(q+1)(q+2)}} - \left[\sum_{q=0}^{\infty}\frac{\Gamma(q-\frac{1}{2})}{q!}\,C^{0,q}_{1,0}\right]^2 - 9\left[\sum_{q=0}^{\infty}\frac{\Gamma(q+\frac{1}{2})}{q!\sqrt{(q+1)(q+2)}}\left(C^{2,q}_{1,0} + S^{2,q}_{0,1}\right)\right]^2 \\
& + \left[\sum_{q=0}^{\infty}\frac{\Gamma(q-\frac{1}{2})}{q!}\,C^{0,q}_{0,1}\right]^2 - 9\left[\sum_{q=0}^{\infty}\frac{\Gamma(q+\frac{1}{2})}{q!\sqrt{(q+1)(q+2)}}\left(C^{2,q}_{0,1} + S^{2,q}_{1,0}\right)\right]^2 .
\end{aligned}
\tag{5.20}
$$

Again, up to this point, the parameter $a(\zeta)$ is arbitrary. A possible choice is to require $a(0) = \langle r^2(0)\rangle^{-1}$ and

$$d_\zeta \hat{a} = \hat{a}\sqrt{\frac{\hat{a}}{2\hat{\alpha}}}\sum_{q=0}^{\infty} A_{1,0}^{0,q}\left[(2q+1)J_{1\,q}^{0} - 2(q+1)J_{1\,q+1}^{0}\right] . \tag{5.21}$$

This is equivalent to requiring $A_{0,0}^{0,1}(0) = 0$ and $d_\zeta A_{0,0}^{0,1} = 0$, which implies $A_{0,0}^{0,1}(\zeta) = 0$ and, from Eq. (5.18), $\langle r^2(\zeta)\rangle = 1/a(\zeta)$. Thus, in a similar fashion as in the 1D case, defining the parameter a by Eq. (5.21) identifies it with the instantaneous value of the reciprocal of the mean-square radius.

VI. CLOSED-FORM SOLUTIONS

In the special case of a hot, emittance-dominated beam, in which the space-charge force is negligible, the model Fokker-Planck equation takes the form

$$(\partial_t + \mathbf{u}\cdot\nabla_x + \mathbf{K}_f\cdot\nabla_u)f = \beta\nabla_u\cdot(f\mathbf{u}) + D\nabla_u^2 f . \tag{6.1}$$

The overbar has been dropped from the distribution function because there is no space-charge-induced turbulence and the stochastic processes generating the transport coefficients are presumed to be from other sources, such as radio-frequency noise or random deviations of the focusing force along the accelerator. If we specialize further to a continuous, linear focusing force for which $K_f = -\omega^2 x$ for the 1D sheet beam, or $K_f = -\omega^2 r$ for the 2D cylindrical beam, then all of the particle orbits are simple-harmonic. Taking β to be constant, we can determine in closed form the first integrals of the Lagrangian subsidiary system associated with Eq. (6.1) and then express the solution for f in terms of these integrals using a procedure documented by Chandrasekhar [30]. This solution is analytic and therefore is advantageous for a number of applications. It can be used to check computer coding of the formalism in Secs. IV and V. Because it provides the basis for determining analytically any desired moment, it can also be used in conjunction with experiments to study the effects on the beam of stochastic processes for which the relaxation rate may be expected to stay constant and the diffusion may be expected to be independent of position and velocity. With the aid of the analytic results, these experiments could be configured to provide measurements of the transport coefficients.

Motivated by these considerations, we proceed to solve Eq. (6.1). We let $p(\mathbf{x},\mathbf{u},t \mid \mathbf{x}_0,\mathbf{u}_0)$ denote the probability of finding a particle with position $\mathbf{x}$ and velocity $\mathbf{u}$ in the range $(\mathbf{x},\mathbf{x}+d\mathbf{x})$, $(\mathbf{u},\mathbf{u}+d\mathbf{u})$, respectively, at time t given it started at $(\mathbf{x}_0,\mathbf{u}_0)$ at $t=0$. Following Ref. [30], we express the solution in terms of the first integrals of the motion

$$\boldsymbol{\xi} = (\mu_1\mathbf{x} - \mathbf{u})e^{-\mu_2 t} , \quad \boldsymbol{\eta} = (\mu_2\mathbf{x} - \mathbf{u})e^{-\mu_1 t} \tag{6.2}$$

and their initial values $(\boldsymbol{\xi}_0, \boldsymbol{\eta}_0)$, in which

$$\mu_1 = -(\beta - \beta_1)/2 , \quad \mu_2 = -(\beta + \beta_1)/2 ,$$
$$\beta_1 = (\beta^2 - 4\omega^2)^{1/2} . \tag{6.3}$$

The solution that tends to $\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0)\,\delta(\boldsymbol{\eta} - \boldsymbol{\eta}_0)$ as $t \to 0$ is

$$p = \left(\frac{\beta_1 e^{\beta t}}{2\pi\sqrt{\Delta}}\right)^d \exp\left\{-\frac{1}{2\Delta}\left[a\,|\boldsymbol{\xi} - \boldsymbol{\xi}_0|^2 + 2h\,(\boldsymbol{\xi} - \boldsymbol{\xi}_0)\cdot(\boldsymbol{\eta} - \boldsymbol{\eta}_0) + b\,|\boldsymbol{\eta} - \boldsymbol{\eta}_0|^2\right]\right\} , \tag{6.4}$$

in which $d = 1$ or 2 for the 1D or 2D case, respectively, and a, b, h, and Δ are functions of time determined from the diffusion coefficient $D(t)$ in the manner

$$a(t) = 2\int_0^t dt'\,D(t')\,e^{-2\mu_1 t'} ,$$
$$b(t) = 2\int_0^t dt'\,D(t')\,e^{-2\mu_2 t'} ,$$
$$h(t) = -2\int_0^t dt'\,D(t')\,e^{-(\mu_1+\mu_2)t'} ,$$
$$\Delta(t) = ab - h^2 . \tag{6.5}$$

The distribution function $f(\mathbf{x},\mathbf{u},t)$ may be determined by multiplying Eq. (6.4) by the initial distribution $f(\mathbf{x}_0,\mathbf{u}_0)$ and integrating over the initial conditions $(\mathbf{x}_0,\mathbf{u}_0)$. To illustrate the procedure, we consider an initial distribution which is Gaussian in both $\mathbf{x}_0$ and $\mathbf{u}_0$, i.e.,

$$f(\mathbf{x}_0,\mathbf{u}_0) = \left(\frac{\alpha_0\omega}{\pi\rho_0}\right)^d \exp\left[-\alpha_0\sum_{i=1}^{d}\left(u_{0i}^2 + \frac{\omega^2}{\rho_0^2}x_{0i}^2\right)\right] . \tag{6.6}$$

Here, as before, we are letting ρ_0 represent the ratio of the initial rms beam size to the matched rms beam size and $\alpha_0 = \alpha(0) = \beta/2D(0)$. With this initial distribution, the integrals can be evaluated in closed form. This calculation, as well as the calculations of the moments delineated below, is entirely straightforward but very tedious, and we used a computer-algebra code as an aid. The result is

$$f(\mathbf{x},\mathbf{u},t) = \left(\beta_1\frac{\alpha_0\omega}{\pi\rho_0\sqrt{C}}\right)^d \exp\left[-\sum_{i=1}^{d}\left(a_{uu}u_i^2 + a_{xu}x_iu_i + a_{xx}x_i^2\right)\right] , \tag{6.7}$$

in which

$$a_{uu} = \frac{\alpha_0}{C}\left[\frac{\omega^2}{\rho_0^2}c_{uu} + (e_2\mu_1 - e_1\mu_2)^2\right] ,$$
$$a_{xu} = -\frac{2\alpha_0}{C}\left\{\frac{\omega^2}{\rho_0^2}c_{xu} + \mu_1\mu_2\left[e_1^2\mu_2 + e_2^2\mu_1 - e_1e_2(\mu_1+\mu_2)\right]\right\} ,$$

$$a_{xx} = \frac{\alpha_0}{\mathcal{C}} \left[\frac{\omega^2}{\rho_0^2} c_{xx} + (e_1 - e_2)^2 \mu_1^2 \mu_2^2 \right] ,$$
$$c_{uu} = 2\alpha_0 \left(a e_1^2 + b e_2^2 + 2h e_1 e_2 \right) + (e_1 - e_2)^2 ,$$
$$c_{xu} = 2\alpha_0 \left[a e_1^2 \mu_1 + b e_2^2 \mu_2 + h e_1 e_2 (\mu_1 + \mu_2) \right] + e_1^2 \mu_1 + e_2^2 \mu_2 - e_1 e_2 (\mu_1 + \mu_2) ,$$
$$c_{xx} = 2\alpha_0 \left(a e_1^2 \mu_1^2 + b e_2^2 \mu_2^2 + 2h e_1 e_2 \mu_1 \mu_2 \right) + (e_1 \mu_1 - e_2 \mu_2)^2 ,$$
$$\mathcal{C} = e_1^2 e_2^2 \left\{ 2\alpha_0 \left[\frac{\omega^2}{\rho_0^2} (2\alpha_0 \Delta + a + b + 2h) + a \mu_1^2 + b \mu_2^2 + 2h \mu_1 \mu_2 \right] + (\mu_1 - \mu_2)^2 \right\} ,$$

where we have used the shorthand notation $e_1 = e^{\mu_1 t}$ and $e_2 = e^{\mu_2 t}$. The distribution function shows clearly that the effect of the linear focusing force is to stratify the phase space into corotating similar confocal ellipsoids and the stochastic processes determine the detailed evolution of their sizes and eccentricity.

The density can be calculated by integrating over the velocity space. The result is

$$n(\mathbf{x}, t) = \frac{1}{L^{3-d}} \left(\frac{1}{\sigma \sqrt{2\pi}} \right)^d \exp \left(-\frac{1}{2\sigma^2} \sum_{i=1}^{d} x_i^2 \right) , \quad (6.8)$$

in which

$$\sigma(t) = \frac{\sigma(0)}{\beta_1} \left[\frac{\omega^2}{\rho_0^2} c_{uu} + (e_2 \mu_1 - e_1 \mu_2)^2 \right]^{1/2} . \quad (6.9)$$

This result gives at once the time dependence of the rms beam size. For the 1D case $\tilde{x} = \sigma$ and for the 2D case $\tilde{r} = \sqrt{2}\sigma$. Likewise, the rms emittance can be calculated and the result for both the 1D and 2D cases is

$$\tilde{\epsilon}(t) = \tilde{\epsilon}(0) \beta_1^{-1} \sqrt{\mathcal{C}(t)} . \quad (6.10)$$

These expressions for the rms quantities are quite simple in the sense that they involve only elementary functions [31]. We have written computer codes to implement the semianalytic formalism for 1D sheet beams [12] and 2D cylindrically symmetric beams developed in the preceding sections and have used the analytic results in conjunction with validating the codes. In so doing, we have mutually confirmed both the codes and the analytic results.

More general analytic expressions for these rms quantities can also be developed. For example, consider a 1D sheet beam that is initially Maxwellian in velocity space but has an arbitrary initial density profile centered on the focusing channel. We may then proceed through the sequence of calculations leading to the moments, beginning with Eq. (6.4) and saving the integration over x_0 for last. In this way we generate expressions for $\sigma^2(t; x_0)$ and $\tilde{\epsilon}^2(t; x_0)$. Finally, integration over x_0 yields the same expressions for $\sigma(t)$ and $\tilde{\epsilon}(t)$ as given in Eqs. (6.9) and (6.10), respectively, in which ω^2/ρ_0^2 may be replaced by $[2\alpha_0 \sigma^2(0)]^{-1}$.

VII. DISCUSSION

The systems of equations for the expansion coefficients, Eqs. (4.12) and (5.11), are equivalent to the coupled Fokker-Planck and Poisson equations. Space charge introduces quadratic terms in the expansion coefficients and the equations must therefore be solved by numerical integration. Because the charge-redistribution phase lasts for a very short time, we may ignore its detailed dynamics and regard the turbulence resulting from charge redistribution to be present at $t = 0$, the time of injection of the beam into the accelerator. A viable strategy for solving the equations would be to truncate the series, solve the truncated series, then increase the number of equations, and solve the larger series. If the solutions substantially agree, then one knows that a sufficient number of terms has been retained in the truncation. Although this procedure involves numerical methods, it should nevertheless be much faster than particle-in-cell codes for most applications. We have developed numerical codes based on Eqs. (4.12) and (5.11), but shall relegate application of the formalism to future papers.

The equations for the expansion coefficients are valid for Fokker-Planck transport coefficients with arbitrary time dependence which must be specified. By way of example, we propose a physically plausible phenomenological model of the diffusion coefficient [12]. We introduce a "diffusive temperature" $T = mD/\beta k_B$ which, in equilibrium, corresponds to the true thermodynamic temperature. We then adopt an exponential model of turbulent heating:

$$D(t) = M^{-1} \beta k_B T(t) = M^{-1} \beta k_B [T_\infty + (T_0 - T_\infty) e^{-\beta_s t}] . \quad (7.1)$$

Starting from temperature T_0, the beam strives to reach a Maxwell-Boltzmann distribution with temperature T_∞ and the heating occurs at the rate $\beta_s \geq \beta$ associated with "strong" turbulence. We would expect $\beta_s \gg \beta$ in a badly rms-mismatched beam because strong turbulence ensues, which then dissipates very rapidly to weak turbulence. Likewise, we would expect $\beta_s \simeq \beta$ in a modestly rms-mismatched beam because the associated turbulence is then relatively weak. The relaxation rate β is left as a free parameter to be specified based on the dominant relaxation mechanism(s) in the beam. The ratio T_∞/T_0 is calculated from the available free energy using Eq. (2.14).

Our model of the turbulence, which incorporates a beam "temperature" and relaxation rate which are independent of both position and velocity, is likely to be most appropriate for particles moving with velocities not much exceeding the thermal velocity, as is the case near thermodynamic equilibrium when Coulomb collisions drive the relaxation [32]. It may therefore be expected to apply to "typical" particles comprising the central region of the beam. In actuality, the relaxation rate will be slower for fast particles because they will have less time to interact with localized field fluctuations. Consequently, because halo particles either move rapidly through the core or orbit around the core, the halo may be expected to thermalize more slowly than the core. This is seen in simulations [2, 3]. However, despite its shortcomings, the model should be useful both for studying the evolution of global irregularities in the beam and for investigating halo generation from the core.

It is clear that a more accurate theory hinges on more accurate transport coefficients. As formulated, our model Fokker-Planck equation is similar to that pertaining to Brownian motion, for which the collision term is

$$\left.\frac{\partial \bar{f}}{\partial t}\right|_c = -\nabla_u \cdot (\mathcal{F}\bar{f}) + \frac{1}{2}\nabla_u \cdot [\nabla_u \cdot (\mathcal{D}\bar{f})] , \qquad (7.2)$$

in which

$$\mathcal{F}(\mathbf{x},\mathbf{u},t) = \frac{\langle \Delta\mathbf{u}\rangle}{\Delta t}, \qquad \mathcal{D}(\mathbf{x},\mathbf{u},t) = \frac{1}{2}\frac{\langle \Delta\mathbf{u}\Delta\mathbf{u}\rangle}{\Delta t} \qquad (7.3)$$

are the dynamical-friction vector and diffusion tensor, respectively. $\Delta\mathbf{u}(\mathbf{x},\mathbf{u},t)$ is an incremental change in the velocity of a single particle occurring over a time Δt, which is long compared to the correlation time of the stochastic space-charge fields seen by the orbiting particle, but short compared to the characteristic time for evolution of the distribution function. To resolve the dynamical behavior of the transport coefficients in a beam which is oscillating and turbulent due to rms mismatch, Δt should be chosen to be a small, but macroscopic, fraction of the particle's orbital period. The averages are taken over the number of particles [32]. More explicitly, if $P_{\Delta t}(\mathbf{x},\mathbf{u},t\,|\,\Delta\mathbf{u})$ is the probability that the velocity $\mathbf{u}$ of a particle at coordinate $\mathbf{x}$ changes by an increment $\Delta\mathbf{u}$ in time Δt, then

$$\langle \Delta\mathbf{u}(\mathbf{x},\mathbf{u},t)\rangle = \int d(\Delta\mathbf{u})\,\Delta\mathbf{u}\,P_{\Delta t}(\mathbf{x},\mathbf{u},t\,|\,\Delta\mathbf{u}) , \qquad (7.4)$$

and likewise for $\langle \Delta\mathbf{u}\Delta\mathbf{u}\rangle$. Equation (7.2) is the same collision term as in Eq. (3.3) with the substitutions

$$\mathbf{F} = -\mathcal{F} + \frac{1}{2}\nabla_u \cdot \mathcal{D}, \qquad \mathsf{D} = \frac{1}{2}\mathcal{D} . \qquad (7.5)$$

One may readily investigate these quantities in conjunction with numerical experiments [33]. Examples of numerical integrations of particle orbits in a fluctuating background indicate that the diffusion coefficient may have a more complicated evolution than simple exponential growth [34] and that the transport coefficients are very sensitive to the form of the fluctuation spectrum [35]. To calculate the averages which generate $\mathcal{F}$ and $\mathcal{D}$, one may study orbits of test particles during the interval Δt in an ensemble of numerical experiments and take ensemble averages of the deflections $\Delta\mathbf{u}$ and $\Delta\mathbf{u}\Delta\mathbf{u}$. This procedure may be successively implemented to cover all of configuration space, velocity space, and time. Alternatively, one may calculate the fluctuation spectrum at each position and time step during the course of a numerical experiment and then construct the coefficients from the spectrum using Eqs. (3.6) and (3.7) as guides. We believe that future work along these lines will prove fruitful. The ultimate goal, of course, is to relate the transport coefficients to accelerator design parameters.

ACKNOWLEDGMENTS

This work was funded in part by the U. S. Army Space and Strategic Defense Command and in part by the Office of Fusion Energy, under the auspices of the U. S. Department of Energy, Contract No. W-31-109-ENG-38. We gratefully acknowledge stimulating discussions with R. L. Gluckstern, S. Habib, R. A. Jameson, J. S. O' Connell, R. Ryne, and T. P. Wangler during our recent stay at the Accelerator Technology Division of Los Alamos National Laboratory. R. A. Jameson also provided helpful comments on the manuscript.

APPENDIX A: PROPERTIES OF THE GAUSS-HERMITE AND GAUSS-LAGUERRE FUNCTIONS

1. Gauss-Hermite functions

The Gauss-Hermite functions are defined as

$$\psi_m(u) = \left(\frac{1}{2^m m!}\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha u^2} H_m(\sqrt{\alpha}u) \qquad (A1)$$

and satisfy the following normalization conditions

$$\int_{-\infty}^{+\infty} \psi_n(u)\,du = \delta(n),$$
$$\int_{-\infty}^{+\infty} \psi_m(u)\,\psi_n(u)\,e^{\alpha u^2}\,du = \sqrt{\frac{\alpha}{\pi}}\,\delta(m-n) , \qquad (A2)$$

in which $\delta(m)$ is the Kronecker delta function, i.e., $\delta(m \neq 0) = 0, \delta(0) = 1$. Other useful properties of these functions are

$$\sqrt{2\alpha}\,u\,\psi_m = \sqrt{m+1}\,\psi_{m+1} + \sqrt{m}\,\psi_{m-1} , \qquad (A3)$$

$$2\alpha\,u^2\,\psi_m = \sqrt{(m+1)(m+2)}\,\psi_{m+2} + (2m+1)\,\psi_m + \sqrt{m(m-1)}\,\psi_{m-2} , \qquad (A4)$$

$$(2\alpha)^{-1/2}\partial_u\,\psi_m = -\sqrt{m+1}\,\psi_{m+1} , \qquad (A5)$$

$$(2\alpha)^{-1}\partial_u^2\,\psi_m = \sqrt{(m+1)(m+2)}\,\psi_{m+2} , \qquad (A6)$$

$$2\alpha\,\partial_t\,\psi_m = -d_t\alpha\left[m\,\psi_m + \sqrt{(m+1)(m+2)}\,\psi_{m+2}\right] , \qquad (A7)$$

$$u\,\partial_u\,\psi_m = -\sqrt{(m+1)(m+2)}\,\psi_{m+2} - (m+1)\,\psi_m . \qquad (A8)$$

In the case of the 1D beam we will also use the Gauss-Hermite functions $\varphi_n(x)$:

$$\varphi_n(x) = \left(\frac{1}{2^n n!}\frac{a}{\pi}\right)^{1/2} e^{-ax^2} H_n(\sqrt{a}x) , \qquad (A9)$$

which, of course, reflect the above properties as well.

2. Gauss-Laguerre functions

The Gauss-Laguerre functions are defined as

$$\phi_q^p(r) = \frac{a}{\pi}\left[\frac{q!}{(|p|+q)!}\right]^{1/2} (ar^2)^{|p|/2}\,e^{-ar^2}\,L_q^{|p|}(ar^2) \qquad (A10)$$

and they satisfy the normalization relations

$$\int_0^\infty r\,\phi_q^p\,dr = \frac{1}{2\pi}\left[\frac{1}{q!\,(|p|+q)!}\right]^{1/2}\frac{|p|}{2}\,\Gamma\left(\frac{|p|}{2}+q\right), \quad \text{(A11)}$$

$$\int_0^\infty r\,e^{ar^2}\,\phi_q^p\,\phi_{q'}^p\,dr = \frac{a}{2\pi^2}\,\delta(q-q'). \quad \text{(A12)}$$

Other properties are

$$\partial_r\,\phi_q^p = \frac{2}{r}\left[-\left(q+\frac{|p|}{2}+1\right)\phi_q^p + \sqrt{(q+1)(q+|p|+1)}\,\phi_{q+1}^p\right], \quad \text{(A13)}$$

$$\partial_t\,\phi_q^p = \frac{d_t\,a}{a}\left[-\left(q+\frac{|p|}{2}\right)\phi_q^p + \sqrt{(q+1)(q+|p|+1)}\,\phi_{q+1}^p\right]. \quad \text{(A14)}$$

APPENDIX B: REDUCTION OF THE FOKKER-PLANCK EQUATION FOR ONE-DIMENSIONAL BEAMS

For one-dimensional sheet beams, the distribution function $\overline{W}(x,u,t)$ satisfies the Fokker-Planck equation

$$(\partial_t + u\partial_x + \overline{K}\partial_u)\overline{W} = \beta\partial_u(u\overline{W}) + D\partial_u^2\overline{W}. \quad \text{(B1)}$$

We first expand $\overline{W}$ in terms of the eigenfunctions $\psi_m(u)$ defined in Appendix A:

$$\overline{W}(x,u,t) = \sum_{m=0}^{\infty} A_m(x,t)\,\psi_m(u). \quad \text{(B2)}$$

Inserting this expansion into the Fokker-Planck equation and using the properties of the Gauss-Hermite functions yields the following system of differential equations for the coefficients $A_m(x,t)$:

$$\partial_t A_m = \frac{d_t\alpha}{2\alpha}\left[mA_m + \sqrt{m(m-1)}A_{m-2}\right] - \sqrt{\frac{m}{2\alpha}}\left[\partial_x - 2\alpha(K_f+\overline{K}_s)\right]A_{m-1} - \sqrt{\frac{m+1}{2\alpha}}\partial_x A_{m+1} - m\beta A_m. \quad \text{(B3)}$$

The coefficients A_m are then expanded in terms of the Gauss-Hermite functions $\varphi_j(x)$ also defined in Appendix A:

$$A_m(x,t) = \sum_{j=0}^{\infty} A_m^j(t)\,\varphi_j(x). \quad \text{(B4)}$$

The space-charge force can be determined from Poisson's equation by straightforward integration with the boundary condition $\overline{K}_s(x=0)=0$:

$$\partial_x\overline{K}_s = \frac{Q^2}{M\varepsilon_0}\bar{n}(x,t) = \frac{Q^2}{LM\varepsilon_0}\sum_{j'=0}^{\infty} A_0^{j'}(t)\,\varphi(x), \quad \text{(B5)}$$

$$\overline{K}_s(x,t) = \frac{Q^2}{2LM\varepsilon_0}\left[A_0^0(t)\,\mathrm{erf}(\sqrt{a}x) - \sqrt{\frac{2}{a}}\sum_{j'=1}^{\infty}\frac{A_0^{j'}(t)}{\sqrt{j'}}\,\varphi_{j'-1}(x)\right]. \quad \text{(B6)}$$

We now define the operator

$$\mathcal{T}[f] = \sqrt{\frac{\pi}{a}}\int_{-\infty}^{+\infty} dx\,e^{ax^2}\,\varphi_n(x)f(x) \quad \text{(B7)}$$

and apply it to Eq. (B3). Since $\mathcal{T}[\varphi_j] = \delta(j-n)$, we obtain, using the properties of the Gauss-Hermite functions listed in Appendix A,

$$\begin{aligned}
\mathcal{T}[A_m] &= A_m^n, \\
\mathcal{T}[\partial_t A_m] &= d_t A_m^n - (2a)^{-1}d_t a\left[nA_m^n + \sqrt{n(n-1)}A_m^{n-2}\right], \\
\mathcal{T}[\partial_x A_{m-1}] &= -\sqrt{2a}\,\sqrt{n}\,A_{m-1}^{n-1}, \\
\mathcal{T}[K_f A_{m-1}] &= -\omega^2\mathcal{T}(xA_{m-1}) \\
&= -(2a)^{-1}\omega^2\left(\sqrt{n}A_{m-1}^{n-1} + \sqrt{n+1}A_{m-1}^{n+1}\right).
\end{aligned} \quad \text{(B8)}$$

Upon applying the operator $\mathcal{T}$ to the space-charge term we obtain

$$\begin{aligned}
\mathcal{T}[\overline{K}_s A_{m-1}] &= \frac{Q^2}{2LM\varepsilon_0}\mathcal{T}\left[\left\{A_0^0\,\mathrm{erf}(\sqrt{a}x) - \sqrt{\frac{2}{a}}\sum_{j'=1}^{\infty}\frac{A_0^{j'}}{\sqrt{j'}}\varphi_{j'-1}\right\}\sum_{j=0}^{\infty}A_{m-1}^j\varphi_j\right] \\
&= \frac{Q^2}{2LM\varepsilon_0}\left\{A_0^0\sum_{j=0}^{\infty}A_{m-1}^j\mathcal{T}[\mathrm{erf}(\sqrt{a}x)\varphi_j] - \sqrt{\frac{2}{a}}\sum_{j'=1}^{\infty}\sum_{j=0}^{\infty}\frac{A_0^{j'}A_{m-1}^j}{\sqrt{j'}}\mathcal{T}[\varphi_{j'-1}\varphi_j]\right\}.
\end{aligned} \quad \text{(B9)}$$

$\mathcal{T}[\mathrm{erf}(\sqrt{a}x)\varphi_j]$ can be evaluated from Eq. (2.20.16.26) of Ref. [36] and is given by

$$\mathcal{T}[\mathrm{erf}(\sqrt{a}x)\varphi_j] = \begin{cases} \dfrac{\sin\left[(j-n)\frac{\pi}{2}\right]}{\pi}\left(\dfrac{1}{j!\,n!}\right)^{1/2}\dfrac{j+n}{j-n}\,\Gamma\left(\dfrac{j+n}{2}\right) & \text{for } j+n \text{ odd} \\ 0 & \text{for } j+n \text{ even.} \end{cases} \tag{B10}$$

$\mathcal{T}[\varphi_{j'-1}\varphi_j]$ can be evaluated from Eq. (7.375.1) of Ref. [37] and is given by

$$\sqrt{\frac{2}{aj'}}\mathcal{T}[\varphi_{j'-1}\varphi_j] = \begin{cases} \dfrac{j+n-j'}{2\pi^2}\left(\dfrac{1}{j'!\,j!\,n!}\right)^{1/2}\Gamma\left(\dfrac{j+n-j'}{2}\right)\Gamma\left(\dfrac{n+j'-j}{2}\right)\Gamma\left(\dfrac{j'+j-n}{2}\right) & \text{for } j+j'+n \text{ odd} \\ 0 & \text{for } j+j'+n \text{ even.} \end{cases} \tag{B11}$$

Setting $j'=0$ in Eq. (B11) gives the same result as Eq. (B10). Thus, if we define

$$I^{j'}_{j\,n} = \begin{cases} \dfrac{j+n-j'}{2\pi^2}\left(\dfrac{1}{j'!\,j!\,n!}\right)^{1/2}\Gamma\left(\dfrac{j+n-j'}{2}\right)\Gamma\left(\dfrac{n+j'-j}{2}\right)\Gamma\left(\dfrac{j'+j-n}{2}\right) & \text{for } j+j'+n \text{ odd,} \\ 0 & \text{for } j+j'+n \text{ even,} \end{cases} \tag{B12}$$

then the contribution from the space-charge term is given by

$$\mathcal{T}[\overline{K}_s A_{m-1}] = \frac{Q^2}{2LM\epsilon_0}\left\{\sum_{j=0}^{\infty} A^j_{m-1}\sum_{j'=0}^{\infty} A^{j'}_0 I^{j'}_{j\,n}\right\} . \tag{B13}$$

Collecting all the terms yields the following differential equation for the coefficients A^n_m:

$$\begin{aligned} d_t A^n_m = {}& \frac{d_t\alpha}{2\alpha}\left[mA^n_m + \sqrt{m(m-1)}A^n_{m-2}\right] + \frac{d_t a}{2a}\left[nA^n_m + \sqrt{n(n-1)}A^{n-2}_m\right] + \sqrt{\frac{a}{\alpha}}\sqrt{n}\left[\sqrt{m}A^{n-1}_{m-1} + \sqrt{m+1}A^{n-1}_{m+1}\right] \\ & -\sqrt{\frac{\alpha}{a}}\omega^2\sqrt{m}\left[\sqrt{n}A^{n-1}_{m-1} + \sqrt{n+1}A^{n+1}_{m-1}\right] - \frac{Q^2}{2LM\varepsilon_0}\sqrt{2\alpha}\sqrt{m}\sum_{j=0}^{\infty} A^j_{m-1}\sum_{k=0}^{\infty} A^k_0\, I^k_{j\,n} - m\beta A^n_m , \end{aligned} \tag{B14}$$

which, after normalization, gives Eq. (4.12).

APPENDIX C: REDUCTION OF THE FOKKER-PLANCK EQUATION FOR TWO-DIMENSIONAL CYLINDRICAL BEAMS

In polar coordinates the distribution function $\overline{W}(r,\theta,u_r,u_\theta,t)$ satisfies the Fokker-Planck equation

$$\left[\partial_t + u_r\,\partial_r + \frac{u_\theta}{r}\,\partial_\theta + \left(\overline{K}_r + \frac{u_\theta^2}{r}\right)\partial_{u_r} + \left(\overline{K}_\theta - \frac{u_r u_\theta}{r}\right)\partial_{u_\theta}\right]\overline{W} = \beta\left(\mathcal{L}_{u_r} + \mathcal{L}_{u_\theta}\right)\overline{W} , \tag{C1}$$

where $\mathcal{L}_{u_r}$ and $\mathcal{L}_{u_\theta}$ are the Fokker-Planck collision operators defined in Eq. (4.5) for the radial and azimuthal velocities, respectively. We first expand $\overline{W}$ in terms of the functions $\psi_m(u_r)$ and $\psi_n(u_\theta)$, which are the eigenfunctions of the operators $\mathcal{L}_{u_r}$ and $\mathcal{L}_{u_\theta}$:

$$\overline{W}(r,\theta,u_r,u_\theta,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n}(r,\theta,t)\,\psi_m(u_r)\,\psi_n(u_\theta) . \tag{C2}$$

Inserting this expansion into the Fokker-Planck equation and using the properties of the Gauss-Hermite functions yields the following set of differential equations for the coefficients $A_{m,n}(r,\theta,t)$:

$$\begin{aligned} \partial_t A_{m,n} = {}& \frac{d_t\alpha}{2\alpha}\left[\sqrt{m(m-1)}A_{m-2,n} + (m+n)A_{m,n} + \sqrt{n(n-1)}A_{m,n-2}\right] - \sqrt{\frac{m}{2\alpha}}\left[-2\alpha\overline{K}_r + \partial_r - \frac{n}{r}\right]A_{m-1,n} \\ & -\sqrt{\frac{n}{2\alpha}}\left[-2\alpha\overline{K}_\theta + \frac{1}{r}\,\partial_\theta\right]A_{m,n-1} - \sqrt{\frac{m+1}{2\alpha}}\left[\partial_r + \frac{n+1}{r}\right]A_{m+1,n} - \sqrt{\frac{n+1}{2\alpha}}\,\frac{1}{r}\,\partial_\theta A_{m,n+1} \\ & +\sqrt{\frac{m}{2\alpha}}\sqrt{(n+1)(n+2)}\,\frac{1}{r}\,A_{m-1,n+2} - \sqrt{\frac{m+1}{2\alpha}}\sqrt{n(n-1)}\,\frac{1}{r}\,A_{m+1,n-2} - (m+n)\beta A_{m,n} , \end{aligned} \tag{C3}$$

where $\overline{K}_r$ and $\overline{K}_\theta$ are the radial and azimuthal components of the acceleration

$$\overline{K}_r = -\omega^2 r - \frac{Q}{M}\partial_r\bar{\Phi}_s , \qquad \overline{K}_\theta = -\frac{Q}{M}\frac{1}{r}\partial_\theta\bar{\Phi}_s , \tag{C4}$$

and $\bar{\Phi}_s$ is the space-charge potential obtained from Poisson's equation

$$\frac{1}{r}\,\partial_r(r\partial_r\bar{\Phi}_s) + \frac{1}{r^2}\,\partial_\theta^2\bar{\Phi}_s = -\frac{Q}{M}\,\bar{n}(r,\theta,t) . \tag{C5}$$

The coefficients $A_{m,n}$ are then expanded in terms of the Gauss-Laguerre functions

$$A_{m,n}(r,\theta,t) = \sum_{p=-\infty}^{+\infty}\sum_{q=0}^{\infty} A_{m,n}^{p,q}\,\phi_q^p(r)\,e^{ip\theta} , \tag{C6}$$

from which we obtain the density

$$\bar{n}(r,\theta,t) = L^{-1}\sum_{p=-\infty}^{+\infty}\sum_{q=0}^{\infty} A_{0,0}^{p,q}\,\phi_q^p(r)\,e^{ip\theta} . \tag{C7}$$

In order to proceed further in reducing the 2D Fokker-Planck equation we need to obtain an expression for the radial and azimuthal accelerations in terms of the expansion coefficients $A_{m,n}^{p,q}$. This requires solving Eq. (C5), where the density is given by Eq. (C7). If we define $\bar{\Phi}_q^p$ as the solution of

$$\nabla^2\bar{\Phi}_q^p = -\phi_q^p\,e^{ip\theta} , \tag{C8}$$

then the space-charge potential corresponding to the density (C7) is given by

$$\bar{\Phi}_s = \frac{Q}{LM}\sum_{p=-\infty}^{+\infty}\sum_{q=0}^{\infty} A_{0,0}^{p,q}\,\bar{\Phi}_q^p \tag{C9}$$

and the radial and azimuthal components of the acceleration are, respectively,

$$\overline{K}_r = -\omega^2 r - \frac{Q^2}{LM\varepsilon_0}\sum_{p=-\infty}^{+\infty}\sum_{q=0}^{\infty} A_{0,0}^{p,q}\,\partial_r\bar{\Phi}_q^p\,e^{ip\theta} , \qquad \overline{K}_\theta = -i\,\frac{Q^2}{LM\varepsilon_0}\sum_{p=-\infty}^{+\infty}\sum_{q=0}^{\infty} A_{0,0}^{p,q}\,\bar{\Phi}_q^p\,\frac{p}{r}\,e^{ip\theta} . \tag{C10}$$

Case (i) $p = q = 0$. Poisson's equation simplifies to

$$\frac{1}{r}\,\partial_r(r\partial_r\bar{\Phi}_0^0) = -\frac{a}{\pi}\,e^{-ar^2} , \tag{C11}$$

which can be integrated directly:

$$\bar{\Phi}_0^0 = \frac{1}{4\pi}\left[\mathrm{Ei}(-ar^2) - C - \ln(ar^2)\right] , \qquad \partial_r\bar{\Phi}_0^0 = \frac{1}{2\pi r}\left(e^{-ar^2} - 1\right) , \tag{C12}$$

where $\mathrm{Ei}(x)$ is the exponential integral function and C is Euler's constant.

Case (ii) $p = 0, q \neq 0$. The space-charge potential is given by

$$\frac{1}{r}\,\partial_r(r\partial_r\bar{\Phi}_q^0) = -\frac{a}{\pi}\,e^{-ar^2}L_q(ar^2) , \tag{C13}$$

which has the solution

$$\bar{\Phi}_q^0 = \frac{1}{4\pi q}\,e^{-ar^2}L_{q-1}(ar^2) = \frac{1}{4aq}\,\phi_{q-1}^0 , \qquad \partial_r\bar{\Phi}_q^0 = \frac{1}{2\pi r}\,e^{-ar^2}\left[L_q(ar^2) - L_{q-1}(ar^2)\right] = \frac{1}{2ar}\left[\phi_q^0 - \phi_{q-1}^0\right] . \tag{C14}$$

Case (iii) $p > 0, q = 0$, *(for* $p < 0$, *replace* p *by* $|p|$*).* The space-charge potential is the solution of

$$\frac{1}{r}\,\partial_r(r\partial_r\bar{\Phi}_0^p) - \frac{p^2}{r^2}\bar{\Phi}_0^p = -\phi_0^p = -\frac{a}{\pi}\left[\frac{(ar^2)^p}{p!}\right]^{1/2} e^{-ar^2} . \tag{C15}$$

The Hankel transform $\mathcal{H}_p[f(r)]$ and its inverse $\mathcal{H}_p^{-1}[F(y)]$ defined as

$$\mathcal{H}_p[f(r)] = \int_0^\infty f(r)\,r\,J_p(yr)\,dr = F(y) , \tag{C16}$$

$$\mathcal{H}_p^{-1}[F(y)] = \int_0^\infty F(y)\,y\,J_p(yr)\,dy = f(r) , \tag{C17}$$

have the property that

$$\mathcal{H}_p\left[\frac{1}{r}\,\partial_r(r\partial_r\Phi) - \frac{p^2}{r^2}\,\Phi\right] = -y^2\,\mathcal{H}_p[\Phi] . \tag{C18}$$

If we apply $\mathcal{H}_p$ to Eq. (C15), we obtain

$$-y^2\,\mathcal{H}_p[\bar{\Phi}_0^p] = -\frac{a}{\pi}\left(\frac{a^p}{p!}\right)^{1/2}\int_0^\infty r^{p+1}\,e^{-ar^2}\,J_p(yr)\,dr = -\frac{1}{2^{p+1}\pi}\left(\frac{a^{-p}}{p!}\right)^{1/2} y^{p-2}\exp\left(-\frac{y^2}{4a}\right) . \tag{C19}$$

$\bar{\Phi}_0^p(r)$ is obtained by applying the inverse Hankel transform to $\mathcal{H}_p[\bar{\Phi}_0^p]$:

$$\bar{\Phi}_0^p(r) = \int_0^\infty \mathcal{H}_p[\bar{\Phi}_0^p]J_p(yr)y\,dy = \frac{1}{2^{p+1}\pi}\left(\frac{a^{-p}}{p!}\right)^{1/2} \times\int_0^\infty y^{p-1}\exp\left(-\frac{y^2}{4a}\right)J_p(yr)\,dy , \tag{C20}$$

which yields

$$\bar{\Phi}_0^p = \frac{1}{4\pi}\left[\frac{(ar^2)^{-p}}{p!}\right]^{1/2}\gamma(p,ar^2), \qquad \partial_r\bar{\Phi}_0^p = \frac{1}{4\pi r}\left[\frac{(ar^2)^p}{p!}\right]^{1/2}\left[2e^{-ar^2} - p(ar^2)^{-p}\,\gamma(p,ar^2)\right] , \tag{C21}$$

where $\gamma(p,x)$ is the incomplete gamma function.

Case (iv) $p>0, q\neq 0$ *(for $p<0$, replace p by $|p|$).* The space-charge potential is the solution of

$$\frac{1}{r}\,\partial_r(r\partial_r\bar{\Phi}^p_q)-\frac{p^2}{r^2}\,\bar{\Phi}^p_q=-\phi^p_q$$
$$=-\frac{a}{\pi}\left[\frac{q!}{(p+q)!}\right]^{1/2}(ar^2)^{p/2}\,e^{-ar^2}\,L^p_q(ar^2)\,. \tag{C22}$$

We apply the Hankel transform to Eq. (C22) to obtain $\mathcal{H}_p[\bar{\Phi}^q_p]$:

$$\mathcal{H}_p[\bar{\Phi}^q_p]=\frac{1}{y^2}\frac{a}{\pi}\left[\frac{q!}{(p+q)!}\right]^{1/2}a^{p/2}\int_0^\infty r^{p+1}\,e^{-ar^2}\,J_p(yr)\,L^q_p(ar^2)\,dr$$
$$=\frac{a}{\pi}\left[\frac{q!}{(p+q)!}\right]^{1/2}\frac{y^{p+2q-2}}{(2a^{1/2})^{p+2q+1}q!}\,\exp\left(-\frac{y^2}{4a}\right)\,. \tag{C23}$$

Then we apply the inverse transform to obtain $\bar{\Phi}^p_q$:

$$\bar{\Phi}^q_p(r)=\frac{1}{4\pi}\frac{1}{\sqrt{q(p+q)}}\left[\frac{(q-1)!}{(p+q-1)!}\right]^{1/2}(ar^2)^{p/2}\,e^{-ar^2}\,L^p_{q-1}(ar^2)$$
$$=\frac{1}{4a}\frac{1}{\sqrt{q(p+q)}}\,\phi^p_q\,,$$
$$\partial_r\bar{\Phi}^p_q=\frac{1}{2\pi r}\left[\frac{q!}{(p+q)!}\right]^{1/2}(ar^2)^{p/2}\,e^{-ar^2}\left[L^p_q(ar^2)-\frac{2q+p}{2q}L^p_{q-1}(ar^2)\right]$$
$$=\frac{1}{2ar}\left[\phi^p_q-\frac{2q+p}{2\sqrt{q(p+q)}}\phi^p_{q-1}\right]\,. \tag{C24}$$

Collecting all the terms we obtain Eqs. (5.8) and (5.9) for the radial and azimuthal components of the acceleration, respectively.

We now define the operator

$$\mathcal{T}[f(r,\theta)]=\frac{\pi}{a}\int_0^{2\pi}d\theta\,e^{-ip\theta}\int_0^\infty dr\,r\,e^{ar^2}\,\phi^p_q(r)\,f(r,\theta) \tag{C25}$$

and apply it to Eq. (C3) where

$$A_{m,n}(r,\theta,t)=\sum_{p'=-\infty}^{+\infty}\sum_{q'=0}^{\infty}A^{p',q'}_{m,n}(t)\,\phi^{p'}_{q'}(r)\,e^{ip'\theta}\,, \tag{C26}$$

and in the expressions for $\overline{K}_r$ and $\overline{K}_\theta$ given by Eqs. (5.8) and (5.9) we use sums over p'' and q''. Since $\mathcal{T}[e^{ip'\theta}\,\phi^{p'}_{q'}]=\delta(p-p')\delta(q-q')$, applying $\mathcal{T}$ to the successive terms in Eq. (C3) yields

$$\mathcal{T}[\partial_t A_{m,n}]=\sum_{p'=-\infty}^{+\infty}\sum_{q'=0}^{\infty}\left\{A^{p',q'}_{m,n}\mathcal{T}\left[\partial_t\phi^{q'}_{p'}\right]+(d_tA^{p',q'}_{m,n})\mathcal{T}\left[\phi^{p'}_{q'}\right]\right\}$$
$$=\frac{d_ta}{a}\left[-\left(\frac{|p|}{2}+q\right)A^{p,q}_{m,n}+\sqrt{q(|p|+q)}A^{p,q-1}_{m,n}\right]+d_tA^{p,q}_{m,n}\,, \tag{C27}$$

$$\mathcal{T}[A_{m-2,n}]=A^{p,q}_{m-2,n}\,,\qquad \mathcal{T}[A_{m,n}]=A^{p,q}_{m,n}\,,\qquad \mathcal{T}[A_{m,n-2}]=A^{p,q}_{m,n-2}\,, \tag{C28}$$

$$\mathcal{T}[\partial_r A_{m-1,n}]=2a^{1/2}\sum_{q'=0}^{\infty}A^{p,q'}_{m-1,n}\left[-\left(\frac{|p|}{2}+q'+1\right)J^p_{q\,q'}+\sqrt{(q'+1)(|p|+q'+1)}J^p_{q\,q'+1}\right]\,, \tag{C29}$$

$$\mathcal{T}[\partial_r A_{m+1,n}]=2a^{1/2}\sum_{q'=0}^{\infty}A^{p,q'}_{m+1,n}\left[-\left(\frac{|p|}{2}+q'+1\right)J^p_{q\,q'}+\sqrt{(q'+1)(|p|+q'+1)}J^p_{q\,q'+1}\right]\,, \tag{C30}$$

$$\mathcal{T}\left[\frac{A_{m-1,n}}{r}\right] = a^{1/2}\sum_{q'=0}^{\infty} A_{m-1,n}^{p,q'} J_{q\,q'}^{p} \;, \quad \mathcal{T}\left[\frac{A_{m+1,n}}{r}\right] = a^{1/2}\sum_{q'=0}^{\infty} A_{m+1,n}^{p,q'} J_{q\,q'}^{p} \;, \tag{C31}$$

$$\mathcal{T}\left[\frac{A_{m-1,n+2}}{r}\right] = a^{1/2}\sum_{q'=0}^{\infty} A_{m-1,n+2}^{p,q'} J_{q\,q'}^{p} \;, \quad \mathcal{T}\left[\frac{A_{m+1,n-2}}{r}\right] = a^{1/2}\sum_{q'=0}^{\infty} A_{m+1,n-2}^{p,q'} J_{q\,q'}^{p} \;, \tag{C32}$$

$$\mathcal{T}\left[\frac{\partial_\theta A_{m,n-1}}{r}\right] = i\,p\,a^{1/2}\sum_{q'=0}^{\infty} A_{m,n-1}^{p,q'} J_{q\,q'}^{p} \;, \quad \mathcal{T}\left[\frac{\partial_\theta A_{m,n+1}}{r}\right] = i\,p\,a^{1/2}\sum_{q'=0}^{\infty} A_{m,n+1}^{p,q'} J_{q\,q'}^{p} \;, \tag{C33}$$

$$\begin{aligned}\mathcal{T}[\overline{K}_r A_{m-1,n}] = {}& -\omega^2 a^{-1/2}\sum_{q'=0}^{\infty} A_{m-1,n}^{p,q'} K_{q\,q'}^{p} \\ & - \frac{\omega_p^2 a^{1/2}}{a_0}\sum_{p'=-\infty}^{+\infty}\sum_{q'=0}^{\infty}\sum_{p''=-\infty}^{+\infty}\sum_{q''=0}^{\infty} A_{0,0}^{p'',q''} A_{m-1,n}^{p',q'} M_{q\,q'\,q''}^{p\,p'\,p''}\,\delta(p-p'-p'') \;,\end{aligned} \tag{C34}$$

$$\mathcal{T}[\overline{K}_\theta A_{m,n-1}] = -i\frac{\omega_p^2 a^{1/2}}{2a_0}\sum_{p'=-\infty}^{+\infty}\sum_{q'=0}^{\infty}\sum_{p''=-\infty}^{+\infty}\sum_{q''=0}^{\infty} A_{0,0}^{p'',q''} A_{m,n-1}^{p',q'} N_{q\,q'\,q''}^{p\,p'\,p''}\,\delta(p-p'-p'') \;. \tag{C35}$$

The constants I, J, K, M, and N are defined as

$$\begin{aligned}I_{q\,q'\,q''}^{p\,p'\,p''} &= \frac{2\pi^3}{a^{5/2}}\int_0^\infty dr\, e^{ar^2}\,\phi_q^p\,\phi_{q'}^{p'}\,\phi_{q''}^{p''} \\ &= \left[\frac{q!\,q'!\,q''!}{(|p|+q)!\,(|p'|+q')!\,(|p''|+q'')!}\right]^{1/2}\int_0^\infty dx\, e^{-2x}\,(x^{1/2})^{|p|+|p'|+|p''|-1}\,L_q^{|p|}(x)\,L_{q'}^{|p'|}(x)\,L_{q''}^{|p''|}(x) \;,\end{aligned} \tag{C36}$$

$$\begin{aligned}J_{q\,q'}^{p\,p'}(p'') &= \frac{2\pi^2}{a^{3/2}}\int_0^\infty dr\, e^{ar^2}\,\phi_q^p\,\phi_{q'}^{p'}\left\{\delta(p'') + [1-\delta(p'')]\frac{|p''|}{2}\left[\frac{(ar^2)^{-|p''|}}{|p''|!}\right]^{1/2}\gamma(|p''|, ar^2)\right\} \\ &= \left[\frac{q!\,q'!}{(|p|+q)!\,(|p'|+q')!}\right]^{1/2}\int_0^\infty dx\, e^{-x}\,(x^{1/2})^{|p|+|p'|-1}\,L_q^{|p|}(x)\,L_{q'}^{|p'|}(x) \\ &\qquad \times\left\{\delta(p'') + [1-\delta(p'')]\frac{|p''|}{2}\left[\frac{x^{-|p''|}}{|p''|!}\right]^{1/2}\gamma(|p''|, x)\right\} \;,\end{aligned} \tag{C37}$$

$$\begin{aligned}J_{q\,q'}^{p} &= J_{q\,q'}^{p\,p}(0) \\ K_{q\,q'}^{p} &= \frac{2\pi^2}{a^{1/2}}\int_0^\infty dr\, r^2\, e^{ar^2}\,\phi_q^p\,\phi_{q'}^{p} \\ &= \left[\frac{q!\,q'!}{(|p|+q)!\,(|p|+q')!}\right]^{1/2}\int_0^\infty dx\, e^{-x}\,x^{|p|+1/2}\,L_q^p(x)\,L_{q'}^p(x) \;,\end{aligned} \tag{C38}$$

$$M_{q\,q'\,q''}^{p\,p'\,p''} = I_{q\,q'\,q''}^{p\,p'\,p''} - \left\{\delta(q'')J_{q\,q'}^{p\,p'}(p'') + [1-\delta(q'')]\frac{2q''+|p''|}{2\sqrt{q''(|p''|+q'')}}I_{q\,q'\,q''-1}^{p\,p'\,p''}\right\} \;, \tag{C39}$$

$$N_{q\,q'\,q''}^{p\,p'\,p''} = 2[1-\delta(p'')]\left\{\delta(q'')J_{q\,q'}^{p\,p'}(p'') + [1-\delta(q'')]\frac{2q''+|p''|}{2\sqrt{q''(|p''|+q'')}}I_{q\,q'\,q''-1}^{p\,p'\,p''}\right\} \;. \tag{C40}$$

Collecting all the terms yields the following set of differential equations for the coefficients $A_{m,n}^{p,q}$:

$$
\begin{aligned}
d_t A^{p,q}_{m,n} = {} & \frac{d_t\alpha}{2\alpha}\left[\sqrt{m(m-1)}A^{p,q}_{m-2,n} + (m+n)A^{p,q}_{m,n} + \sqrt{n(n-1)}A^{p,q}_{m,n-2}\right] \\
& + \frac{d_t a}{2a}\left[(|p|+2q)A^{p,q}_{m,n} - 2\sqrt{q(|p|+q)}A^{p,q-1}_{m,n}\right] \\
& + \sqrt{\frac{a}{2\alpha}}\sqrt{m}\sum_{q'=0}^{\infty} A^{p,q'}_{m-1,n}\left\{\left[|p|+2(q'+1)+n\right]J^{p}_{q\,q'} - 2\sqrt{(q'+1)(q'+|p|+1)}J^{p}_{q\,q'+1}\right\} \\
& + \sqrt{\frac{a}{2\alpha}}\sqrt{m+1}\sum_{q'=0}^{\infty} A^{p,q'}_{m+1,n}\left\{\left[|p|+2(q'+1)-(n+1)\right]J^{p}_{q\,q'} - 2\sqrt{(q'+1)(q'+|p|+1)}J^{p}_{q\,q'+1}\right\} \\
& + \sqrt{\frac{a}{2\alpha}}\sum_{q'=0}^{\infty} J^{p}_{q\,q'}\left\{\sqrt{m(n+1)(n+2)}A^{p,q'}_{m-1,n+2} - \sqrt{n(n-1)(m+1)}A^{p,q'}_{m+1,n-2}\right. \\
& \qquad\qquad \left. - i\,p\left(\sqrt{n}A^{p,q'}_{m,n-1} + \sqrt{n+1}A^{p,q'}_{m,n+1}\right)\right\} \\
& - \sqrt{\frac{2\alpha}{a}}\sqrt{m}\,\omega^2\sum_{q'=0}^{\infty} A^{p,q'}_{m-1,n}K^{p}_{q\,q'} \\
& - \sqrt{\frac{2\alpha}{a}}\frac{a}{2a_0}\omega_p^2\sum_{p'=-\infty}^{+\infty}\sum_{q'=0}^{\infty}\sum_{p''=-\infty}^{+\infty}\sum_{q''=0}^{\infty} A^{p'',q''}_{0,0}\,\delta(p-p'-p'')\left(2\sqrt{m}A^{p',q'}_{m-1,n}M^{p\,p'\,p''}_{q\,q'\,q''} - i\sqrt{n}A^{p',q'}_{m,n-1}N^{p\,p'\,p''}_{q\,q'\,q''}\right) \\
& - (m+n)\beta A^{p,q}_{m,n}\,,
\end{aligned} \tag{C41}
$$

which after normalization gives Eq. (5.11).

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